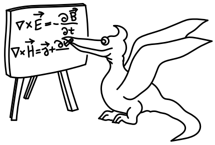

Serial: Symmetry and Linear Algebra

In the previous part of the series we expanded our mathematical toolkit, which enabled us to explore and describe some more complicated systems. We saw that when several particles oscillate, or a particle oscillates in more than one dimension, more types of oscillations in the same system can occur. For every extra dimension or every extra particle, there is a corresponding number of equations, which determine the solution we search for. We have also seen how we can determine the ratio of amplitudes and phase difference between oscillations in different directions.

Now, we will try to derive a stronger form of these results – instead of solving the equations for oscillations separately, we will solve all the equations at once, using the formalism of linear algebra. This formalism needs to be introduced first.

Linear Algebra

The purpose of linear algebra is to describe the behaviour of linear transformations of vectors. Let's analyse the different concepts in previous statement.

Vectors are collections of numbers, with some defined operations. They are usually indicated by an arrow above the letter or by using bold font; here we will describe them as $\mathbf{v}$. The components of the vectors are written as the symbol for the vector together with a lower index that is associated with the given component, i.e. the first component of vector $\mathbf{u}$ is marked as u_1. Sometimes, a letter instead of a number can be used as the index, for example associating the component with a given Cartesian axis, so we can speak of the x-component of vector $\mathbf{t}$, marked as t_x. In our case, vectors will be collections of coordinates of all particles for all possible directions of motion, e.g. if we were studying a system consisting of three particles, with two particles fixed to a plane and one particle free to move through a whole space, our vector would have 7 components.

We can define vector addition in terms of the vector components as follows

$$\mathbf{t} = \mathbf{u} + \mathbf{v} : \forall n : t_n = u_n + v_n \,,$$

where n is taken from the possible indices of $\mathbf{u}$, which are the same set as indices of $\mathbf{v}$ and hence also $\mathbf{t}$. Next, we define scalar multiplication for scalar a

$$\mathbf{v} = a\mathbf{u} : \forall n : v_n = au_n \,.$$

Vector subtraction can be interpreted as a combination of vector addition and scalar multiplication

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v} \,.$$

Last important operation, which we define between vectors, is the scalar product (different from scalar multiplication). This product reduces two vectors to a single scalar, and for vectors $\mathbf{u}$ and $\mathbf{v}$ it is defined as follows

$$s = \mathbf{u} \cdot \mathbf{v} = \sum_n u_n v_n \,,$$

where n is again taken from the available indices of $\mathbf{u}$ and $\mathbf{b}$. If a scalar product of two vectors is equal to zero, we say that the vectors are *perpendicular* to each other.

We can notice that vector addition will correspond to the superposition of two types of oscillations. Hence, the vectors replicate the properties of solutions of linear differential equations, which we use to represent the oscillating systems.

Vector Basis

If we imagine vector as a position in Cartesian system of coordinates, it is clear that the vector can be decomposed into a sum of other vectors, with all the vectors perpendicular to each other. This type of decomposition of the vector is called the basis decomposition. In our algebraical notation, we could write

$$\mathbf{u} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3\mathbf{e}_1 + 5\mathbf{e}_2 \,,$$

where we defined basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. We can use scalar product to check that the basis vectors are in fact perpendicular

$$\mathbf{e}_1 \perp \mathbf{e}_2 \iff \mathbf{e}_1 \cdot \mathbf{e}_2 = 0 \,.$$

The choice of the basis is not unique however – we could have chosen a different basis set

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \,, \mathbf{e}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \,,$$

which still includes two perpendicular vectors, and we could have written

$$\mathbf{u} = \mathbf{e}_1 + 4\mathbf{e}_2 \,.$$

Generally, it is useful to choose basis vectors so that they have a unit length, i.e. they satisfy the condition

$$\forall n : \mathbf{e}_n \cdot \mathbf{e}_n = 1 \,,$$

which is not satisfied by the previous example.

Algebraically, the decomposition into an orthonormal basis (that is, basis set of perpendicular vectors, each of unit length) can be realized by consequent scalar products of the vector with basis vectors taken from the basis set. The scalar products determine the components of the vector in the direction of the given basis vector, i.e. we can write

$$\mathbf{u} = (\mathbf{u} \cdot \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{u} \cdot \mathbf{e}_2)\mathbf{e}_2$$

We should note that such basis decomposition can be done for an arbitrary number of basis vectors. Therefore, we do not need to limit ourselves to three dimensions, as we are used to in the geometrical interpretation of vectors.

Linear Transformations of Vectors

Operations such as scalar multiplication or scalar product are linear operations, but they are only a few examples of all possible linear transformations that can be done with vectors. For example, we could create a vector $\mathbf{v}$ with components $v_1 = u_2 + u_1, v_2 = 0$, where $\mathbf{u}$ is a different vector. Generally, we can define a linear transformation T of vector $\mathbf{u}$ as a new vector $\mathbf{v} = T(\mathbf{u})$, with the transformation satisfying

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})\,.$$

Let's now try to split the transformation into elementary steps. We know that vectors can be split into sum of basis vectors, multiplied by the compenents of the vector – that is what we called the basis decomposition. For example

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2\,.$$

So, linear transformation T of $\mathbf{u}$ satisfies

$$T(\mathbf{u}) = \mathbf{v} = T(\mathbf{e}_1)u_1 + T(\mathbf{e}_2)u_2\,.$$

How can we find the components of the new vector $\mathbf{v}$? We need to find its projection onto the basis vectors, i.e.

$$\begin{aligned} v_1 &= \mathbf{e}'_1 \cdot \mathbf{v} = \mathbf{e}'_1 \cdot T(\mathbf{e}_1)u_1 + \mathbf{e}'_1 \cdot T(\mathbf{e}_2)u_2\,, \\ v_2 &= \mathbf{e}'_2 \cdot \mathbf{v} = \mathbf{e}'_2 \cdot T(\mathbf{e}_1)u_1 + \mathbf{e}'_2 \cdot T(\mathbf{e}_2)u_2\,, \end{aligned}$$

where basis vectors $\mathbf{e}$ and $\mathbf{e}'$ can, but need not to be from the same basis set. Hence, we can see that in order to determine the components of the transformed vector, we only need to know the components of the original vector (u_1 and u_2) and a set of coefficients, which can be labeled as

$$m_{11} = \mathbf{e}'_1 \cdot T(\mathbf{e}_1)\,,\, m_{12} = {\mathbf{e}_1}' \cdot T(\mathbf{e}_2)\,,\, m_{21} = \mathbf{e}'_2 \cdot T(\mathbf{e}_1)\,,\, m_{22} = \mathbf{e}'_2 \cdot T(\mathbf{e}_2)\,.$$

These coefficients are *independent* of a specific vector $\mathbf{u}$. They only reflect the properties of the transformation T and of the chosen basis set(s). These coefficients can be organised into an object we call a matrix.

Matrix Algebra

When we compare the expressions for components of the matrix m_{ij} and the basis decomposition of a vector, we can see that the matrix can be interpreted as a vector that consists of other vectors. Transformation of vector $\mathbf{u}$ can then be seen as application of the scalar product between the vectors inside the matrix and the vector $\mathbf{u}$. This observation can be used to define the matrix product, basic building block of matrix algebra. In order to understand the matrix product, lets explicitly write the matrix as collection of vectors

$$M = \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix}\,.$$

This matrix acts on vector **i**, resulting into a new transformed vector **v**

$$\mathbf{v} = M\mathbf{u} = \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix} \mathbf{u} = \begin{pmatrix} \mathbf{m}_1 \cdot \mathbf{u} \\ \mathbf{m}_2 \cdot \mathbf{u} \end{pmatrix} = \begin{pmatrix} m_{11}u_1 + m_{12}u_2 \\ m_{21}u_1 + m_{22}u_2 \end{pmatrix} .$$

Usually we write the matrix as a collection of the components. In that case, the vectors that constitute the matrix need to be written horizontally – the need for this swap will be explained later. Then, the matrix M is

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} .$$

Matrix product can then be defined without reference to the vectors of the matrix. The i component of vector $\mathbf{v} = M\mathbf{u}$ is defined as

$$v_i = (M\mathbf{u})_i = \sum_j M_{ij} u_j ,$$

where indices j runs over all indices of vector $\mathbf{u}$. Here, we can notice first condition on the objects in matrix product – the number of dimensions of vector $\mathbf{u}$ has to be equal to the number of columns of matrix M.

This definition of matrix product can be readily extended to products of two matrices $A = MB$ as

$$A_{ij} = \sum_k M_{ik} B_{kj} .$$

A specific example of the matrix product follows

$$\begin{pmatrix} 1 & 3 & 2 \\ 5 & -1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 5 & -7 \\ 20 & 4 \end{pmatrix} .$$

Another important matrix operation is the so called transposition, which effectively swaps the rows and columns of a matrix. Transpose of matrix M is labeled M^T. A simple example follows

$$\begin{pmatrix} 1 & 3 & 2 \\ 5 & -1 & 3 \end{pmatrix}^T = \begin{pmatrix} 1 & 5 \\ 3 & -1 \\ 2 & 3 \end{pmatrix} .$$

In terms of the components of the matrix, transposition can be written as

$$(M_{ij})^T = M_{ji} .$$

Vectors are a specific case

$$\begin{pmatrix} 1 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 \\ 2 \end{pmatrix} .$$

So, the original equation for multiplication of vector $\mathbf{u}$ by a matrix M should be correctly written as

$$M\mathbf{u} = \begin{pmatrix} \mathbf{m}_1^T \\ \mathbf{m}_2^T \end{pmatrix} \mathbf{u} .$$

Furthermore, we can notice that scalar product of two vectors can be written as matrix product of the transposed and original vector, i.e.

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} \, .$$

Finally, we define component-wise addition and scalar multiplication for matrices, similarly to vectors

$$A = B + M : A_{ij} = B_{ij} + M_{ij} \, ,$$
$$sA : (sA)_{ij} = sA_{ij} \, .$$

These are the fundamentals of matrix algebra. We will use these to solve equations, which mix the number of degrees of freedom in a non-trivial way. In order to solve these equations completely, however, we need an additional piece of knowledge from the linear algebra. This will be the concept of eigenvectors and eigenvalues.

Eigenvectors and Eigenvalues

Square matrices transform vectors into new vectors with the same number of components. It is therefore possible that the transformed vector is the same vector as the original, up to a scalar factor. Such vector is called the eigenvector of the given matrix, and the corresponding scalar multiple is called the eigenvalue of the matrix for the given eigenvector. These variables represent important properties of the matrix. Algebraically, the eigenvector $\mathbf{v}$ of matrix M is defined by equation

$$M\mathbf{v} = \lambda \mathbf{v} \, ,$$

where λ is the corresponding eigenvalue. A square matrix of dimension n can have up to n different eigenvectors, each with a corresponding eigenvalue. How can we find these eigenvectors? We can try to guess the eigenvector based on some properties of the system such as the symmetry of the system (see later), or we can determine the eigenvector based on the corollary of the previous equation

$$(M - \lambda I)\mathbf{v} = \mathbf{0} \, ,$$

where I is the so called unit matrix – a square matrix with components equal to zero everywhere except at the diagonal, where the components are equal to 1. We can quickly check that any valid matrix product of unit matrix and a vector or another square matrix leaves the other vector (or matrix) unchanged. The equation above has either a trivial solution with all components of $\mathbf{v}$ equal to zero, or the matrix $M - \lambda I$ contains column vectors, which can be combined in such a way that the resulting vector is zero. But this means that the vectors are linearly dependent, i.e. at least one of the vectors can be expressed as linear combination of the others. An important result from the theory of linear equations is that in such a case, the variable called the determinant of the matrix is equal to zero.

Determinant can be determined for any matrix, but we will mainly use the determinant for 2x2 matrices. For such matrices, the determinant can be calculated from the components of the matrix as

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow \quad |M| = ad - bc \, ,$$

where $|M|$ stands for the determinant of the matrix M. Next, we will use determinant of a diagonal matrix, which is simply the product of the diagonal elements. As a generalization, when a matrix consists of several blocks of matrices lying on a diagonal, the determinant of the overall matrix is the product of the determinants of the individual blocks. For example

$$M = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & f \end{pmatrix} \quad \Rightarrow \quad |M| = ef(ad - bc)\,,$$

where we used that the determinant of 1x1 matrix is equal to the only component of the matrix.

Now, we can explore an example calculation of eigenvector and eigenvalue. Consider matrix

$$M = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} .$$

In order to find the eigenvalues, we need to solve equation

$$\begin{aligned} |M - \lambda I| &= 0\,, \\ \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} &= 0\,, \\ (1-\lambda)^2 - 4 &= 0\,, \\ (1-\lambda)^2 &= 4\,, \\ 1 - \lambda &= \pm 2\,, \\ \lambda &= 1 \pm 2\,,\ \lambda \in \{3, -1\}\,. \end{aligned}$$

As we know the eigenvalues λ, we can find the corresponding eigenvectors

$$\begin{aligned} (M - \lambda I)\mathbf{v} &= 0\,, \\ \lambda = -1: \begin{pmatrix} 1-(-1) & 2 \\ 2 & 1-(-1) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}\,, \\ \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}\,. \end{aligned}$$

The solution of this system of equations is clear – $v_1 = -v_2$ and the eigenvector is for example

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

We say for example, because the eigenvector can be multiplied by an arbitrary scalar number, without changing its behaviour under the trasformation by M. The important quantity is the ratio between the components of the eigenvector, not the absolute value.

As a note – we could wonder why did we not simply solve the separate rows of equation $(M - \lambda I)\mathbf{v} = 0$? The problem is that in the given set of n equations, we have $n+1$ unknowns – n components of the eigenvector and the eigenvalue. The result would be the same in the end, we would obtain equation with the same component of $\mathbf{v}$ on both sides, so we could divide by this component, provided it is different from zero. However, it is usually a lot simpler to find the determinant of the matrix, because we immediately get the eigenvalue, which usually has some physical significance as well. The fact that we are still solving system of n equations and $n + 1$ unknowns presents itself in the fact that the eigenvector can be multiplied by arbitrary scalar multiplier.

Normal Modes

Enough with the obscure mathematics, let's focus on physics now. In the following examples, we will illustrate the usefulness of the developed techniques – we will be able to describe the oscillations of non-trivial oscillators. The first example includes two masses, interconnected by a spring and connected to a wall, while oscillating in vertical direction only. The second example features two particles, connected by a spring and connected to walls on both sides, free to move in a plane.

Two Masses Underneath the Ceiling

Consider the following setup: first spring with spring constant k is attached to the unmoving ceiling on one end and to a mass m on the other end. Second spring, also with spring constant k, is attached to the first mass on one end and to a second (but otherwise identical) mass m on the other end. The equilibrium position of the system (when no oscillations occur) takes place when the gravitational forces balance the tension in the springs. We will not determine this equilibrium position here, application of basic results from statics can be used to get its characteristics. We will only consider the small displacements from this equilibrium position. Let us label the displacement of the first mass as x_1 and the displacement of the second mass as x_2. Assume that both displacement can be described as oscillating, i.e.

$$x_1(t) = \mathrm{Re}\left(Ae^{\mathrm{i}\omega t}\right) ,$$

$$x_2(t) = \mathrm{Re}\left(Be^{\mathrm{i}\omega t}\right) .$$

Any phase difference between the oscillations can be expressed as part of the constant B, which can be written as

$$B = |B|e^{\mathrm{i}\varphi} ,$$

where φ is the phase difference. Then

$$x_2(t) = \mathrm{Re}\left(|B|e^{\mathrm{i}(\omega t+\varphi)}\right) .$$

First spring is elongated by x_1 compared to the equilibrium position, the second spring is elongated by $x_2 - x_1$. The tension in the second spring is given as

$$F_2 = -k(x_2 - x_1) .$$

In the first spring, the tension is

$$F_1 = -kx_1 - F_2 ,$$

as the force in the second spring needs to be balanced by the force in the first spring. Due to the Newton's second law

$$F_1 = m\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = -kx_1 + k(x_2 - x_1) ,$$

$$F_2 = m\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = -k(x_2 - x_1) .$$

Applying Fourier substitution

$$-m\omega^2 x_1 = -2kx_1 + kx_2 ,$$

$$-m\omega^2 x_2 = kx_1 - kx_2 .$$

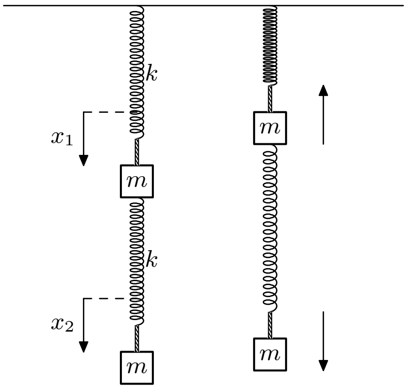

Fig. 1: The geometry of the problem is sketched on the left, including definitions of x_1 and x_2. On the right, a snapshot during the oscillations of one of the normal modes is displayed. The arrows indicate the direction of motion.

This equation can be written as matrix equation – we are trying to find oscillations of x_1 and x_1, which are independent, so we can interpret them as different dimensions of the motion. Specifically, lets define vector $\mathbf{x}$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

Then,

$$\omega^2 M\mathbf{x} = K\mathbf{x}\,,$$

where the matrix K is

$$K = \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix} ,$$

and matrix M is

$$M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

and both sides of the equation were multiplied by (-1). Hence, our system of equations is represented by a single matrix equation

$$\omega^2 \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

This leads to

$$\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ,$$

therefore

$$(K - \omega^2 M)\mathbf{x} = \mathbf{0}\,.$$

This is a variant of the equation, which needs to be solved in order to find the eigenvalues of a matrix. Again, we need to find value of ω such that the determinant of the matrix in the brackets is zero, $|K - \omega^2 M| = 0$. The determinant can be determined as

$$\begin{vmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{vmatrix} = (2k - m\omega^2)(k - m\omega^2) - k^2 = 0\,,$$

and hence

$$\begin{aligned} 2k^2 - 2km\omega^2 - km\omega^2 + m^2\omega^4 - k^2 &= 0\,, \\ m^2\omega^4 - 3mk\omega^2 + k^2 &= 0\,. \end{aligned}$$

Dividing by m^2 and defining $\omega_0^2 = \frac{k}{m}$ leads to

$$\omega^4 - 3\omega_0^2\omega^2 + \omega_0^4 = 0\,.$$

This biquadratic equation can be solved as a quadratic equation for ω^2

$$\omega^2 = \frac{3\omega_0^2 \pm \sqrt{9\omega_0^4 - 4\omega_0^4}}{2}\,.$$

Therefore, we discover two possible oscillation frequencies

$$\omega = \omega_0\sqrt{\frac{3}{2} \pm \sqrt{\frac{5}{4}}}\,.$$

What is the ratio of the amplitudes of oscillations, and the phase difference? To find these values, we need to also find the eigenvectors of the matrix. First row of the matrix equation is

$$m\omega^2 x_1 = 2kx_1 - kx_2\,.$$

Since we already know the eigenvalue, we can substitute this value in to get

$$\begin{aligned} m\omega_0^2\left(\frac{3}{2} \pm \sqrt{\frac{5}{4}}\right)x_1 &= 2kx_1 - kx_2\,, \\ \omega_0^2\left(\frac{3}{2} \pm \sqrt{\frac{5}{4}}\right)x_1 &= 2\omega_0^2 x_1 - \omega_0^2 x_2\,. \end{aligned}$$

We can divide this equation by ω_0^2 and by a factor $e^{\mathrm{i}\omega t}$, which is included in both x_1 and x_2, and therefore

$$\begin{aligned} \left(\frac{3}{2} \pm \sqrt{\frac{5}{4}}\right)A &= 2A - B\,, \\ \left(-\frac{1}{2} \pm \sqrt{\frac{5}{4}}\right)A &= -B\,, \\ \frac{B}{A} &= \frac{1}{2} \mp \sqrt{\frac{5}{4}}\,. \end{aligned}$$

The ratio of the amplitudes is therefore different for different frequencies. For higher frequency

$$\omega = \sqrt{\frac{3}{2} + \sqrt{\frac{5}{4}}}\sqrt{\frac{k}{m}}$$

the system oscillates so that the ratio of the amplitude of the second mass to the amplitude of the first mass is

$$\frac{B}{A} = \frac{1}{2} - \frac{\sqrt{5}}{2} \,.$$

This ratio is negative, meaning that in any moment of the oscillation, the masses are moving in opposite directions. The lower frequency

$$\omega = \sqrt{\frac{3}{2} - \sqrt{\frac{5}{4}}}\sqrt{\frac{k}{m}}$$

corresponds to the situation with ratio of amplitudes

$$\frac{B}{A} = \frac{1}{2} + \frac{\sqrt{5}}{2}$$

and the masses move in the same direction (but at different speeds) during the oscillations. In terms of phase difference, we could have written

$$-1 = e^{i\pi}$$

and therefore determine that the masses oscillate exactly in anti-phase, a result which we obtained anyway.

Let's take a moment to realize our achievement. We found that for system of **two** particles, there are **two** special frequencies, at which the dynamical equations for oscillations can be satisfied. These frequencies correspond to two types of oscillations, characterised by components A and B of the oscillation vector. These types of oscillations are called normal modes.

You could argue that we have only described a very specific case of the motion, and that there surely exist a more complex motion the system can exhibit. The key strength of the normal modes description lies in the linearity of the dynamical equations. Since the equations are linear, any superposition of the two eigenvectors we found is a valid motion of the system as well. Algebraically, if we have an oscillation vector $\mathbf{A}_1$ such that

$$\omega^2 M\mathbf{A}_1 = K\mathbf{A}_1$$

and another $\mathbf{A}_2$ for which

$$\omega^2 M\mathbf{A}_2 = K\mathbf{A}_2 \,,$$

then, for any scalar factors a and b,

$$\omega^2 M\left(a\mathbf{A}_1 + b\mathbf{A}_2\right) = K\left(a\mathbf{A}_1 + b\mathbf{A}_2\right)$$

and hence even the vector $a\mathbf{A}_1 + b\mathbf{A}_2$ is the solution of the dynamic equations. So, besides the two specific types of oscillations, we also discovered infinitely many types of motions, all of which can be interpreted as *superpositions* of the normal modes. This characteristic truly captures the ease of description of linear systems – in order to describe whole classes of motion, we only need a small number of parameters.

Elementary String

Now, let's turn our attention to the second example. Let there be two particles of equal mass m. First particle is connected to a wall by a spring with spring constant k. The wall intersects the origin of the system of coordinates. The second particle is connected to a second wall (parallel to the first wall) at point $\mathbf{R}$, which is perpendicular to planes of both walls. The spring constant of the spring connecting the second mass is also k. Finally, both particles are connected together by one more spring with spring constant k. The particles are free to move in plane perpendicular to the plane(s) of the wall(s).

The solution of this example is presented here in a somewhat shorter form, in order to allow you to fill the gaps yourselves. Let's start by finding the equilibrium position. Let the position of the first particle be determined by vector $\mathbf{r}_1$ and the position of the second particle by vector $\mathbf{r}_2$. The force acting on the first particle is

$$\mathbf{F}_1 = -k\mathbf{r}_1 + k\left(\mathbf{r}_2 - \mathbf{r}_1\right) .$$

The force acting on the second particle is

$$\mathbf{F}_2 = -k(\mathbf{r}_2 - \mathbf{R}) + k\left(\mathbf{r}_1 - \mathbf{r}_2\right) .$$

In the equilibrium position, the net forces are zero, and hence

$$\mathbf{r}_1 = \frac{1}{3}\mathbf{R} ,$$

$$\mathbf{r}_2 = \frac{2}{3}\mathbf{R} .$$

We will label these positions as $\mathbf{r}_{1,0}$ and $\mathbf{r}_{2,0}$. Small displacements from these positions will be labeled as $\mathbf{x}_1$ and $\mathbf{x}_2$, so that $\mathbf{r}_1 = \mathbf{r}_{1,0} + \mathbf{x}_1$, and similarly for the second particle. For these small oscillations, the forces on the particles are

$$\mathbf{F}_1 = -k\mathbf{x}_1 + k\left(\mathbf{x}_2 - \mathbf{x}_1\right) ,$$

$$\mathbf{F}_2 = -k\mathbf{x}_2 + k\left(\mathbf{x}_1 - \mathbf{x}_2\right) .$$

The Newton's second law states that

$$\mathbf{F}_1 = m\frac{\mathrm{d}^2\mathbf{r}_1}{\mathrm{d}t^2} ,$$

$$\mathbf{F}_2 = m\frac{\mathrm{d}^2\mathbf{r}_2}{\mathrm{d}t^2} .$$

Since $\mathbf{r}_{1,0}$ and $\mathbf{r}_{2,0}$ are constant vectors

$$\mathbf{F}_1 = m\frac{\mathrm{d}^2\mathbf{x}_1}{\mathrm{d}t^2} ,$$

$$\mathbf{F}_2 = m\frac{\mathrm{d}^2\mathbf{x}_2}{\mathrm{d}t^2} .$$

Again, we will assume that the system oscillates. From the Fourier substitution, we can derive $\mathbf{F}_1 = -m\omega^2\mathbf{r}_1$, and similarly for the second particle. In the matrix form, our system of equations becomes

$$\begin{pmatrix} m\omega^2 - 2k & 0 & k & 0 \\ 0 & m\omega^2 - 2k & 0 & k \\ k & 0 & m\omega^2 - 2k & 0 \\ 0 & k & 0 & m\omega^2 - 2k \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} ,$$

where x_{11} is the first component of vector $\mathbf{x}_1$ etc. The determinant of this matrix is not readily obtainable, because it is not a diagonal matrix. We can notice however that only components of indices 1 and 2 are mixed together, respectively. If we regroup the components in our vector so that these components are next to each other, we obtain the following matrix equation

$$\begin{pmatrix} m\omega^2 - 2k & k & 0 & 0 \\ k & m\omega^2 - 2k & 0 & 0 \\ 0 & 0 & m\omega^2 - 2k & k \\ 0 & 0 & k & m\omega^2 - 2k \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

This matrix consists of two matrices on the diagonal, and therefore we know how to find the determinant, i.e.

$$D = \begin{vmatrix} m\omega^2 - 2k & k & 0 & 0 \\ k & m\omega^2 - 2k & 0 & 0 \\ 0 & 0 & m\omega^2 - 2k & k \\ 0 & 0 & k & m\omega^2 - 2k \end{vmatrix} ,$$

$$D = \begin{vmatrix} m\omega^2 - 2k & k \\ k & m\omega^2 - 2k \end{vmatrix}^2 = \left(\left(m\omega^2 - 2k\right)^2 - k^2\right)^2 .$$

Putting $D = 0$ leads to

$$\begin{aligned} k^2 &= \left(m\omega^2 - 2k\right)^2 , \\ k &= \pm\left(m\omega^2 - 2k\right) , \\ (2 \pm 1)k &= m\omega^2 . \end{aligned}$$

Let

$$\omega_0 = \sqrt{\frac{k}{m}} ,$$

which leads to

$$\omega = \sqrt{2 \pm 1}\omega_0 .$$

Thus, we found only two frequencies, even though there are total of 4 normal modes. This means that some of the modes have identical frequencies. By substitution of the eigenvalue into the matrix equation, we can find the eigenvectors. Let's start with $\omega = \omega_0$

$$\begin{pmatrix} m\frac{k}{m} - 2k & k & 0 & 0 \\ k & m\frac{k}{m} - 2k & 0 & 0 \\ 0 & 0 & m\frac{k}{m} - 2k & k \\ 0 & 0 & k & m\frac{k}{m} - 2k \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

Again, we encounter the fact that the two blocks of the matrix are independent. That is to say, the first two components of the eigenvector are not mixed with the other two components by the matrix, and vice versa. Hence, we can find two eigenvectors corresponding to this single frequency (as we expected), for example

$$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}.$$

For $\omega = \sqrt{3}\omega_0$, the eigenvectors are

$$\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\end{pmatrix}.$$

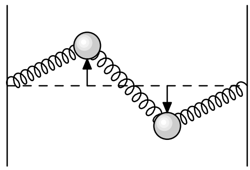

Fig. 2: Indicated direction of motion for the particles during normal mode oscillation.

The Role of Symmetry

Many oscillations problems have some form of symmetry. For example, in the previous problem we could exchange the coordinates used for the description of the first and second particle without changing the problem – we would get exactly the same set of dynamic equations. We therefore say that the system is symmetric under the exchange of particles.

The symmetry can be defined in terms of a matrix – such matrix S will have the property that (using the notation from the previous example)

$$S\begin{pmatrix}x_{11}\\x_{21}\\x_{12}\\x_{22}\end{pmatrix} = \begin{pmatrix}x_{21}\\x_{11}\\x_{22}\\x_{12}\end{pmatrix},$$

which corresponds to the exchange of particles. The determination of such matrix is relatively straightforward – each component of the new vector can be related to a single component of the original vector, which is chosen by the matrix S. Hence

$$S = \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

We expect that the modes of the system will somehow obey this symmetry. It turns out that a general rule applies. When an oscillating system obeys a certain symmetry, than the eigenvectors of the oscillations are at least partially given as eigenvectors of the symmetry matrix. The proof of this statements is not so obvious, and is not derived here. For the more curious among you, the symmetry can be more rigorously defined as invariance of Hamiltonian of the system under the application of the symmetry operation. You can try to decipher the consequences of this statement in the context of Noether's theorem, if you know it. However, for the solution of the problems in this series this is not necessary.

What will be necessary is the ability to determine the matrix that applies the symmetry operation (most commonly particle exchange) and the ability to find the eigenvectors of the given symmetry matrix. In the previous case, we can quickly check that the eigenvectors from the previous example are indeed eigenvectors of S with eigenvalues ± 1. The modes are therefore either symmetric or antisymmetric under the particle exchange.

The advantage of searching for eigenvectors of symmetry matrices lies in the simplicity of these matrices compared to the dynamic matrices, which drive the oscillations. That means, when we obtain eigenvectors of the symmetry matrix, we can simply plug these into the dynamic equations to get the eigenvalues of oscillations.

Incomplete Symmetries

It is possible for a system to have a certain symmetry that does not completly determine the behaviour of the system. For example, consider a ball moving in a valley created by extending a parabola lying in the xy plane (given for example by condition $y = x^2$) into the z direction – the equation remains $y = x^2$, independent of z. Clearly, the symmetry given by mirror plane through yz plane is a symmetry of the system, i.e. symmetry given by $x \to -x$. Let the position of the ball be given by vector $\mathbf{r} = \begin{pmatrix} x & y & z \end{pmatrix}$, then the symmetry matrix is

$$S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Notice that the block of two lower rows and two columns on the right form an identity matrix in components y and z. This means, that the symmetry defined by S does not pose any restrictions on these components – y and z can take any value (in other words, applying S to $\mathbf{r}$ leaves these components unchanged). This means that this symmetry can eliminate up to one degree of freedom from our dynamic equation. Hence, only one eigenvector of oscillations stems from this symmetry of the system. However, since the eigenvectors of oscillations are perpendicular, we can use the eigenvectors of symmetry to eliminate the necessary degrees of freedom and have an easier time when solving for the eigenvectors of oscillations.

When substituting eigenvectors of symmetry matrix for the eigenvectors of oscillations, we must be careful when the symmetry is incomplete – if the given component of the eigenvector is not restricted by the symmetry, we need to substitute a general number for this component. This has to be done because even though the component is not restricted by the symmetry we found, it might be restricted by the symmetry of laws of dynamics itself. For example, conservation of momentum can impose restrictions on the components, which are otherwise unrestricted by the symmetry under exchange of particles.

Infinite Number of Oscillators

We learned how to solve systems containing a certain finite number of oscillators. How do we approach systems where the number of oscillators goes effectively to infinity? In such a case, we can sometimes change its description to a description using continuum variables, leading to the description of the wave phenomenon. Some elementary properties of waves, such as dispersion relation or superposition will be discussed in the next episode of the series.

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