Problem III.E ... game of discharges

13 points; průměr 6,85; řešilo 39 studentů

Charge the object by rubbing it and then measure the dependence of its self-discharge on time. Determine the electrical conductivity of air. Consider that the magnitude of the charge varies according to:

$$Q = Q_0 \mathrm{e}^{-\frac{\sigma}{\varepsilon} t} ,$$

where Q_0 is the initial charge, ε is the permeability of the air, and σ is the conductivity we are looking for.

Hint: Hang a small metallic object (e.g. a nut) on a thin, long filament. Then take a straw, rub it to charge it, and transfer some of the charges to the object. It should begin to repel away from the straw. Afterwards, you can determine the product of the charges and the conductivity from their relative distances.

Jarda tried to measure the charge for so long that he changed the entire problem to measure the conductivity.

Introduction and theoretical basis

The experiment can be measured as indicated in the hint. There is no dependence on the shape of the objects in the relation for the discharging rate. We will therefore continue to assume that neither the shape nor the position of the objects plays any role in the loss of charge. We will use two hex nuts from the Merkur construction set, which we will tie to a long thread, and then throw over a rod and place the hex nuts at the same height. Electric charge can be transferred to a plastic straw by rubbing it against a paper handkerchief and touching the hex nuts with the straw will transfer part of the charge to them. The straw can be charged once more and placed between the two nuts, which should now repel away from each other and from the straw which we have secured in its place. One of the hex nuts may electrostatically "stick" to the straw while the other hex nut begins to be repulsed. That will not affect our measurements of the charge's dependence on time as the shape of the objects is irrelevant.

Although the procedure explained in the previous paragraph works, we ended up using only one hanging hex nut in our solution (see figure below), which we let repel off the straw. We placed this as close to the original location as possible. The position of the hex nut will be recorded with the camera for several minutes, during which the charge on the straw and the hex nut will spontaneously discharge.

We will now focus on the geometry of the experiment and how to use it to obtain information about the charge.

Since the deviation of the hex nut from the perpendicular has only been small, we can consider $\tan\alpha \approx \sin\alpha \approx \alpha$ and $\cos\alpha \approx 1$. Then $\alpha = d/l$, where l is the hinge's length and d is the transverse distance of the centers of the hex nut and the straw.

The hex nut is subject to a repulsive electrostatic force F_e, gravitational force F_G and the tension force of the hinge T. In the small-angle approximation of the α, F_e acts horizontally and matches the tension force $T\alpha$, while the gravitational is equal to the vertical component of $T\cos\alpha \approx T$. We can express T from one equation and plug it into the other to get $F_\mathrm{e} = mg\alpha$.

Now comes the difficulty of how to express the electric force F_e. We cannot simply use Coulomb's law because we do not have point sources of charge. We will try to calculate the electric force using the relation $F_\mathrm{e} = EQ_\mathrm{m}$, where Q_m is the charge on the hex nut and E is the electric intensity induced by the charge on the straw. It can be estimated as a function of the distance from the center of the straw as

$$E = \frac{\lambda}{2\pi\varepsilon d}\,,$$

where λ is the length density of the charge on the straw, which we consider to be homogeneous (we do not have a better estimate). We also consider the straw to be a long thin wire.

We can express the dependence between the charge and the distance d from the equalities of the forces as

$$mg\alpha = \frac{\lambda}{2\pi\varepsilon d}Q_\mathrm{m} \quad \Rightarrow \quad \frac{2\pi\varepsilon mg}{l}d^2 = \lambda Q_\mathrm{m}\,.$$

At the measurement's beginning, there was a charge Q_{m_0} on the hex nut and a charge density λ_0 on the straw. Thus, we can write the right-hand side of the equation as

$$\lambda Q_\mathrm{m} = \lambda_0 \mathrm{e}^{-\sigma t/\varepsilon} Q_{\mathrm{m}0} \mathrm{e}^{-\sigma t/\varepsilon} = \lambda_0 Q_{\mathrm{m}0} \mathrm{e}^{-2\sigma t/\varepsilon}\,.$$

The dependence of the mutual distance d on time is then expressed as

$$d = \sqrt{\frac{l\lambda_0 Q_{\mathrm{m}0}}{2\pi\varepsilon mg}}\mathrm{e}^{-\sigma t/\varepsilon}\,.$$

Thus, the distance should decrease exponentially over time.

Measured values and input processing

The distance was determined using a camera stationed at a fixed location. It took a snapshot of the position of the hex nut relative to the straw every 10 s (see figure below). We hung the hex nut on a long thread and noted where its stable position was. We charged the straw by rubbing it. Touching the hex nut transferred some of the charge on it. The hex nut began to repel away from the straw. We positioned the straw to be right next to the stable position of the hex nut so that when the nut was discharged, the thread was vertical and the hex nut was in its original stable position.

The mass of the hex nut used is approximately $m = 0.4\,\mathrm{g}$, and the hinge's length was $l = 76\,\mathrm{cm}$. The distance of the opposite parallel sides of the hex nut is $r = 0.6\,\mathrm{cm}$.

In the program *Geogebra*, we measured the distance between the centers of the hex nut and the straw in each snapshot. We measured it in relative units. Using the dimension of the hex nut in the real world and relative units, we were able to convert the distances in the snapshots to centimeters using the trinomial

$$d = \frac{r}{r'}d'\,,$$

where r is the dimension of the hex nut in centimeters and the dashed quantities are given in relative units in the snapshots. We chose these units to make the conversion trivial, i.e. $r' = 0.6\,\mathrm{j}$, which corresponds to the aforementioned diameter of the hex nut $r = 0.6\,\mathrm{cm}$.

Three (usable) measurements were made. At a certain point in the convergence of the hex nut to the straw, we saw a change in behavior – the hex nut rapidly approached the straw and "stuck" to it – an attractive force began to act. We estimated the error of the individual distances in the photo to be 0.1 cm, however for graphical illustration we give the values in the table to three significant digits. We did not plot the error bars because their value would be the same everywhere and because we only estimated the error.

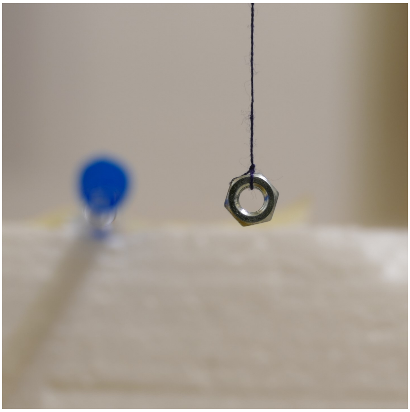

Fig. 1: Pictures of experiment execution.

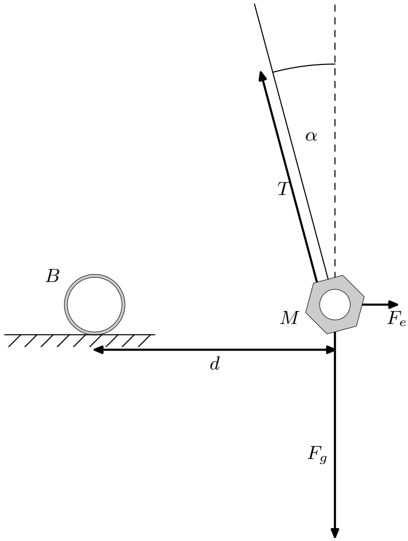

Fig. 2: Outline with important elements. The sizes of the quantities are not to scale.

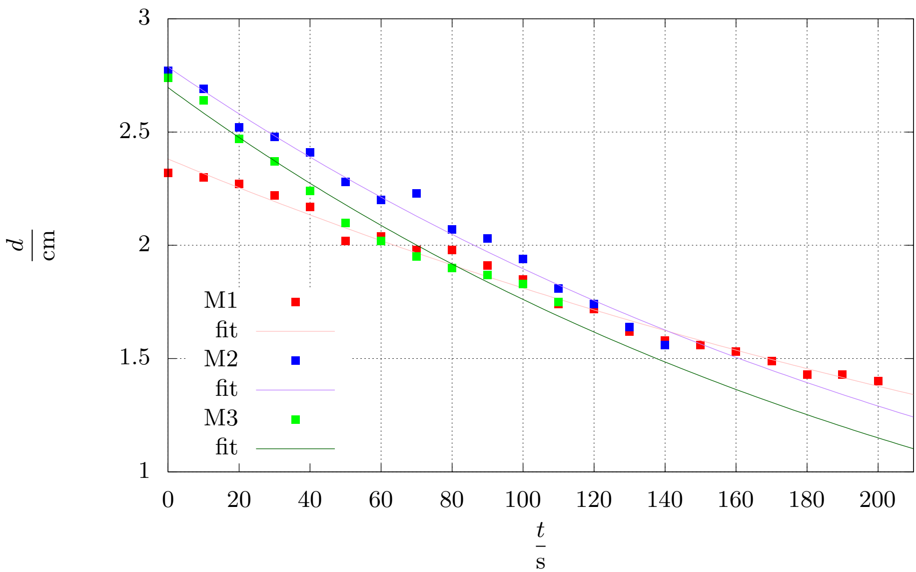

Fig. 3: Distance dependence between the straw and the hex nut on time

Tab. 1: Dependence of the distance between the hex nut and the straw on time. The values are given after conversion from the relative units. To illustrate the decrease, we put them to three significant digits.

$\frac{t}{\mathrm{s}}$	$\frac{d}{\mathrm{cm}}$	$\frac{d}{\mathrm{cm}}$	$\frac{d}{\mathrm{cm}}$
0	2.32	2.77	2.74
10	2.30	2.69	2.64
20	2.27	2.52	2.47
30	2.22	2.48	2.37
40	2.17	2.41	2.24
50	2.02	2.28	2.10
60	2.04	2.20	2.02
70	1.98	2.23	1.95
80	1.98	2.07	1.90
90	1.91	2.03	1.87
100	1.85	1.94	1.83
110	1.74	1.81	1.75
120	1.72	1.74	0
130	1.62	1.64	0
140	1.58	1.56	0
150	1.56	1.45	0
160	1.53	1.10	0
170	1.49	0.51	0
180	1.43	0	0
190	1.43	0	0
200	1.40	0	0

We plot all three dependencies on the graph and interpolate them with the exponential. For measurement number 2 we will ignore the last three data points because the system has clearly begun to behave according to different laws. In measurements 2 and 3, we have fewer data points because the hex nut has been pulled to the block earlier than in measurement 1.

In the program *Gnuplot*, we have interpolated the dependencies with exponentials and shown their equations in the form

$$y = A\mathrm{e}^{Bt}\,. \tag{1}$$

The coefficients represent the numerical value of parameters in equation 1. Errors of these coefficients were also calculated by *Gnuplot*.

Tab. 2: Fitting parameters. The symbol Δ denotes the errors of the corresponding quantities.

Parameter	Measurement 1	Measurement 2	Measurement 3	Average
$\frac{A}{\mathrm{cm}}$	238	290	270	
$\frac{\Delta A}{\mathrm{cm}}$	2	10	3	
$\frac{B}{\mathrm{s}^{-1}}$	$-2.7 \cdot 10^{-3}$	$-3.8 \cdot 10^{-3}$	$-4.3 \cdot 10^{-3}$	$-3.6 \cdot 10^{-3}$
$\frac{\Delta B}{\mathrm{s}^{-1}}$	$0.1 \cdot 10^{-3}$	$0.1 \cdot 10^{-3}$	$0.2 \cdot 10^{-3}$	$0.5 \cdot 10^{-3}$

Calculating the average for the value of the parameter A does not make much sense since this parameter depends on the initial charge, which is different for each measurement. On the contrary, B should be similar in all cases and equal to fraction $-\sigma/\varepsilon$. The conductivity of air is thus given by

$$\sigma = -B\varepsilon = (3.2 \pm 0.4) \cdot 10^{-14}\,\mathrm{S{\cdot}m^{-1}}\,.$$

We put $\varepsilon = \varepsilon_0 = 8.85 \cdot 10^{-12}\,\mathrm{F{\cdot}m^{-1}}$, since the permittivity of air is close to the permittivity of vacuum, and this rounding causes a negligible error.

The relative error of the conductivity σ is the same as the relative error of the parameter B since the two quantities differ only by the multiplication by the constant ε, which has an error of an order of magnitude smaller and can be neglected. The error of parameter B, in the table above in the column, labeled *Average*, is determined as the standard deviation of the arithmetic mean of parameter B.

Discussion

The experiment itself was quite difficult to perform. The charge was not always transferred from the straw to the hex nut in the way we needed, and the repulsion did not occur. Often, the hex nut was attracted to the straw during the whole measurement. In only a few cases did the experiment succeed as intended. However, even during these measurements, the attractive forces prevailed and the hex nut eventually adhered to the straw. Once the charge on the straw was partially discharged, the electrostatic induction caused the rest of the charge to rearrange. If the straw and the hex nut were charged with a charge of the same sign, the hex nut on the side facing the straw would induce a charge of the opposite sign. Although its magnitude may have been smaller than the magnitude of the opposite charge on the hex nut, it was closer to the straw, so the electrostatic force in the opposite direction prevailed.

The dimensions of objects play a large role in designing the appropriate model for a given situation. According to the relation from the assignment, the shape of the body is not contributing to the result. However, we do not know anything about the distribution of charge on the body, the magnitude, and the location of the electrostatic force.

In our model, we assumed that the straw is a uniformly charged long wire. It is however very unlikely that we would charge the straw even approximately homogeneously over any longer portion of it. In addition, we immediately transferred a portion of the charge to the hex nut. It was not too far from the straw during the experiment, so we cannot neglect the cylindrical shape of the straw. At the same time, the straw is also not "very long" compared to the scale of the experiment, and we should probably account for its finite dimensions, but that would be mathematically challenging.

A similar problem arises with the hex nut. In the proposed model, we have considered it as a point charge with its position at the center. But even the hex nut has dimensions that are not negligible relative to the scale of the experiment, so this model is quite bold. At the same time, it is made of conductive material so the charge can rearrange itself, and the vector of electrostatic force changes as well.

According to the problem statement, we had to measure the time dependence of the discharge. In other words, we were to find the dependency of the total charge on time. However, as part of our experiment, we were unable to separate the individual charges in the $Q_\mathrm{m}\lambda$ product. But according to the theory in the problem statement, both charges are discharged at the same rate, so the time dependence of $\sqrt{Q_\mathrm{m}\lambda}$ also satisfies the assignment. According to our model, $\sqrt{Q_\mathrm{m}\lambda}$ is directly proportional to distance d, so by plotting d on the graph, we have actually plotted the dependence of $\sqrt{Q_\mathrm{m}\lambda}$ on time, just in different units.

From the measured values, we have observed that the charge caused by friction is lost in a matter of minutes. In ordinary life, these phenomena do not limit us in any meaningful way.

Let us compare our measured value of air conductivity $(3.2 \pm 0.4) \cdot 10^{-14}\,\mathrm{S{\cdot}m^{-1}}$ with the values given on the Internet. Wikipedia[1] lists a range from $1 \cdot 10^{-15}\,\mathrm{S{\cdot}m^{-1}}$ to $1 \cdot 10^{-9}\,\mathrm{S{\cdot}m^{-1}}$ at room temperature, so it depends very much on other air parameters such as humidity. Elsewhere[2] we can find a range from $3 \cdot 10^{-15}\,\mathrm{S{\cdot}m^{-1}}$ to $8 \cdot 10^{-15}\,\mathrm{S{\cdot}m^{-1}}$. Let us return to the previous discussion of our model and realize that the force between the straw and the hex nut is probably not proportional to d^{-1}, as we have assumed. The force will depend on the size of the straw, and even though this dependence is approximated on a given scale by a power function, the value in the exponential is likely higher (in the absolute value because at finite dimensions, the straw behaves at least a little more like a point source). The magnitude of this exponent will be reflected in the measured conductivity in direct proportion, meaning that the conductivity is that much larger how much the exponential is.

Another inaccuracy is due to the assumption $\alpha = d/L$, which is quite difficult to meet, as it requires precise placement of the straw relative to the hex nut. Also, the definition of the distance d over which the electrostatic force acts, as the distance between the centers of the two bodies may not be correct because in the hex nut, the charge can move freely and we cannot be sure that the point of application of the force is exactly at the center.

[1] `https://en.wikipedia.org/wiki/Electrical_resistivity_and_conductivity`.

[2] `https://www.thoughtco.com/table-of-electrical-resistivity-conductivity-608499`.

Despite all these inaccuracies, we can state that we have determined the conductivity of air on the right order of magnitude and we have an actual idea of how large (or rather small, as we have measured something of the order of magnitude of $1 \cdot 10^{-14}\,\mathrm{S{\cdot}m^{-1}}$, though of course it depends on the units used) it is. We have also found out that air is an insulator since its conductivity is small.

Conclusion

We have set up the experiment according to the hint in the problem statement and measured the distance of the hex nut from the straw as a function of time. The determination of the charge is then, according to the chosen model, proportional to this distance. After fitting the data with an exponential function, we have determined the conductivity of air under normal conditions as $(3.2 \pm 0.4) \cdot 10^{-14}\,\mathrm{S{\cdot}m^{-1}}$.

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