Problem III.P ... absurd pendulum

9 points; průměr 5,88; řešilo 67 studentů

What phenomena can affect the measurement of gravitational acceleration using a pendulum? Estimate how many valid digits your result would have to contain to measure them. Consider also the phenomena that you usually neglect.

Kačka was wondering what she could write in the discussion.

Real changes in gravitational acceleration

The first group of phenomena we will discuss is the real changes in gravitational acceleration, which are caused by effects observable with a sophisticated instrument. The first difference from the tabulated value $9.81\,\mathrm{m{\cdot}s^{-2}}$ are the local deviations based on the location.

Variations by latitude The Earth is not a precise sphere, but rather so-called *geoid*, which is an irregular figure; however, we will consider it to be an ellipsoid with an equatorial radius $R_\mathrm{r} = 6\,378\,\mathrm{km}$ and a polar radius $R_\mathrm{p} = 6\,357\,\mathrm{km}$ in the first approximation.[1] Therefore, even the gravitational accelerations will be different from a free-fall acceleration as the distance from us to the center of the Earth is different depending on the latitude. If we consider all the mass to be concentrated in the center of the Earth, we can calculate free-fall acceleration in both cases. The Earth's mass is $M_\mathrm{E} = 5.973\,6 \cdot 10^{24}\,\mathrm{kg}$, so the free-fall acceleration at the pole will be

$$g_\mathrm{pole} = G\frac{M_E}{R_p^2} = 6.674 \cdot 10^{-11}\,\mathrm{m^3{\cdot}kg^{-1}{\cdot}s^{-2}} \cdot \frac{5.973\,6 \cdot 10^{24}\,\mathrm{kg}}{(6\,357\,\mathrm{km})^2} \doteq 9.865\,\mathrm{m{\cdot}s^{-2}}$$

and at the equator

$$g_\mathrm{eq} = G\frac{M_Z}{R_p^2} = 6.674 \cdot 10^{-11}\,\mathrm{m^3{\cdot}kg^{-1}{\cdot}s^{-2}} \cdot \frac{5.973\,6 \cdot 10^{24}\,\mathrm{kg}}{(6\,378\,\mathrm{km})^2} \doteq 9.801\,\mathrm{m{\cdot}s^{-2}}\,.$$

Since the gravitational force is the net force of the free fall and the centrifugal force of the Earth's rotation, gravitational acceleration will have different values at different latitudes. At the pole, there is no centrifugal acceleration, while at the equator with an equatorial radius $R = 6\,378\,\mathrm{km}$ the centrifugal acceleration is of the magnitude

$$a_\mathrm{c} = \omega^2 R = \frac{4\pi^2}{(1\,\mathrm{day})^2} \cdot 6378\,\mathrm{km} \doteq 0.03\,\mathrm{m{\cdot}s^{-2}}\,.$$

Therefore, the latitude affects the acceleration in the order of hundredths of a $\mathrm{m{\cdot}s^{-2}}$, so we have to measure to at least three significant digits to be able to observe this effect.

Changes by altitude We will calculate the changes by the altitude in the same way as we did for the change by latitude using a model of mass concentrated in the center, for two points on the equator. One point lies at the sea level with a radius $R_\mathrm{r} = 6\,378\,\mathrm{km}$ and the second point is at the peak of Cayambe, which is almost on the equator and has an altitude 5 790 m. a. s. l.,[2] so its radius is $R_\mathrm{Cay} = 6\,383.79\,\mathrm{km}$. The gravitational acceleration at sea level is from the

[1] https://is.muni.cz/el/1441/podzim2007/ZS1BP_IVZ1/um/02.Tvar_a_velikost_Zeme.pdf

[2] https://cs.wikipedia.org/wiki/Cayambe

previous example and is equal to $g_0 = 9.80\,\mathrm{m{\cdot}s^{-2}} - 0.03\,\mathrm{m{\cdot}s^{-2}} = 9.77\,\mathrm{m{\cdot}s^{-2}}$, the gravitational acceleration at Cayambe is

$$\begin{aligned} g_{\mathrm{Cay}} &= G\frac{M_Z}{R_{\mathrm{Cay}}^2} - \omega^2 R_{\mathrm{Cay}} = \\ &= 6.674\cdot 10^{-11}\,\mathrm{m^3{\cdot}kg^{-1}{\cdot}s^{-2}} \cdot \frac{5.9736\cdot 10^{24}\,\mathrm{kg}}{6\,383.79\,\mathrm{km}^2} - \frac{4\pi^2}{1\,\mathrm{day}^2}\cdot 6\,383.79\,\mathrm{km} \doteq \\ &\doteq 9.78\,\mathrm{m{\cdot}s^{-2}} - 0.03\,\mathrm{m{\cdot}s^{-2}} = \\ &= 9.75\,\mathrm{m{\cdot}s^{-2}}\,, \end{aligned}$$

so the difference is in the second decimal place, again. For smaller altitude differences (those of the order of hundreds of meters) the difference will be in the third decimal place.

The subsoil influence In addition to latitude and altitude, the subsoil and the surroundings' relief also influence the result. According to the geology textbook [3] changes in the gravity field due to underground alterations, such as a subsurface oil deposit will be of the order $10^{-6}\,\mathrm{m{\cdot}s^{-2}}$, so we would have to measure to seven significant digits to see its effect.

The cosmic surroundings We have already neglected all the effects of the Earth and the location of a reference point on it. Let us look further, namely into the space. The objects here are very far away, but they are also very massive, and some of them have a significant influence on the Earth. We will calculate the changes in gravitational acceleration due to these objects as the difference of the gravitational force onto a point at the center of the Earth in both positions. We calculate the gravitational acceleration according to the formula

$$g = G\frac{M}{r^2}\,,$$

where $G = 6.67{\cdot}10^{-11}\,\mathrm{m^3{\cdot}s^{-2}{\cdot}kg^{-1}}$ is the gravitational constant, M is the mass of the particular object, and r is its distance from the Earth. The results for the selected bodies are given in the table 1.

Tab. 1: Influence of cosmic objects on gravitational acceleration

object	$\frac{M}{\mathrm{kg}}$	smaller distance	larger distance	$\frac{\Delta g}{\mathrm{m{\cdot}s^{-2}}}$
Moon	$7,35\cdot 10^{22}$	365 033 km	407 241 km	$7\cdot 10^{-6}$
Sun	$1,99\cdot 10^{30}$	147 098 074 km	152 097 701 km	$4\cdot 10^{-4}$
Mars	$6,42\cdot 10^{23}$	0,5 AU	2,5 AU	$6\cdot 10^{-9}$
Jupiter	$1,90\cdot 10^{27}$	4,2 AU	6,2 AU	$2\cdot 10^{-7}$
Pluto	$1,30\cdot 10^{23}$	38 AU	40 AU	$3\cdot 10^{-13}$

The values of masses and distances used in this table are from Wikipedia[4], where for the Sun and the Moon we have used the distances in the perihelion and aphelion or perigee and apogee,

[3] https://is.muni.cz/el/1431/podzim2007/Z0135/um/Uvod_06_Tihove_pole.pdf
[4] https://cs.wikipedia.org/

respectively, while for the planets, we use the sum and the difference of the mean semi-axis of the specific planet and the Earth.

We can now compare the influence of these cosmic objects with the gravitational influence of small and very close objects. For a human of mass $M = 80\,\mathrm{kg}$ at a distance $r = 1\,\mathrm{m}$ is the gravitational acceleration approximately $\Delta g = 5 \cdot 10^{-9}\,\mathrm{m{\cdot}s^{-2}}$ and for a truck with a mass $M = 40\,\mathrm{t}$ at distance $r = 50\,\mathrm{m}$ is the gravitational acceleration approximately $\Delta g = 1 \cdot 10^{-9}\,\mathrm{m{\cdot}s^{-2}}$, comparable to the effect of a human or Mars.

Tidal forces In the previous paragraph, we calculated the effect of cosmic bodies on the point at the center of the Earth. However, these forces are already included in the motion of the Earth in the solar system through trajectory changes. However, what does influence the measurement on the surface is the tidal force caused by the fact that the gravitational force from each object is not the same on the surface of the Earth as at its center. Therefore, the magnitude of the tidal force is proportional to the gradient of the gravitational force. We can observe the tidal acceleration on the line passing through the centers of the two objects

$$\frac{2GMr}{R^3}\,,$$

where M is the mass of the particular object, r is the radius of the Earth, R is the distance of the Earth from the object, and M is the object's mass. After inserting the values for the Moon, we get $\Delta g \approx 5 \cdot 10^{-7}\,\mathrm{m{\cdot}s^{-2}}$ and for the Sun $\Delta g \approx 1 \cdot 10^{-6}\,\mathrm{m{\cdot}s^{-2}}$.[5] Although these forces are small, they are still observable in tides and variations in height according to the moon's phase. This effect is an order of magnitude smaller than the effect of given gravitational differences on the center of the Earth, especially for more distant objects like the Sun (due to the dependence on its third power). Thus, we can assume that the influence of other objects will also be of one order of magnitude smaller than their direct influence on the center of the Earth. If the objects rotate around each other, the effect of centrifugal force, which is approximately half the magnitude of the previously mentioned force, must also be accounted for.[6]

The systematic problems of the mathematical pendulum

We have discussed the actual changes in gravitational acceleration, and now, let us look at the systematic errors of the measurement using a mathematical pendulum. For our example we will consider that our pendulum is made of an iron sphere of mass $m = 1\,\mathrm{kg}$, which is tethered to a steel rod in such a way that its center of gravity is $l = 2\,\mathrm{m}$.

The mathematical pendulum, the parameter accuracy In the mathematical pendulum model, we have the following formula for its period

$$T = 2\pi\sqrt{\frac{l}{g}}\,.$$

From the period measurement, we can calculate the acceleration of gravity as

$$g = 4\pi^2\frac{l}{T^2}\,.$$

[5] https://en.wikipedia.org/wiki/Tidal_force#Sun,_Earth,_and_Moon
[6] https://cs.wikipedia.org/wiki/Slapov%C3%A1_s%C3%ADla

The question now arises as to what accuracy we would have to measure the rod's period and length to achieve the desired precision. Using the propagation of uncertainty formula will help us determine that the relative error of the rod length and acceleration will be the same. Thus, if we wanted to measure the deviations of the gravitational acceleration of the order of $10^{-9}\,\mathrm{m{\cdot}s^{-2}}$, we would need to achieve a relative accuracy of 10^{-10}, meaning that we would have to measure the suspension with an accuracy of $10^{-10}\,\mathrm{m}$, which is on the order of the size of an atom. Accuracy of the time measurement is even more important here because according to the propagation of uncertainty, just this error would cause the relative error in the measurement of the gravitational acceleration to be twice the relative error of the period measurement. If we were to achieve a measurement accuracy for g of the order of $10^{-5}\,\mathrm{m{\cdot}s^{-2}}$ (theoretically possible for length if we could measure micrometers), we would need to have a relative period measurement accuracy of $5\cdot 10^{-6}$, which for our pendulum with a period of approximately $2.8\,\mathrm{s}$ gives an uncertainty in the measurement of time approximately $10^{-5}\,\mathrm{s}$. For stopwatch measurements with a possible measurement accuracy $0.1\,\mathrm{s}$ it would mean measuring 10^4 swings, which would take about 8 hours with a given period. In such a long time, the pendulum would probably already have stopped as a result of other effects. To achieve the required measurement accuracy for 100 swings (5 minutes), we would need to measure to an accuracy of at least $0.001\,\mathrm{s}$.

Mathematical pendulum, changes of length To have the length stabilized to micrometers, we need to consider the phenomena that could change it, namely thermal expansion, and elastic deformation. Changes of the length due to linear thermal expansion is calculated as

$$\Delta l = l_0 \alpha \Delta T\,,$$

where l_0 is the original length, ΔT is the temperature change and α is is the coefficient of linear expansion. Its magnitude can be found, for example in tables[7] and we see that its value is, e.g., for steel $11\cdot 10^{-6}\,\mathrm{K^{-1}}$. Thus, we would need to keep the temperature stable to tenths of a degree, while for polyethylene it is more than ten times larger, so we would need to keep the temperature stable to a hundredth of a degree.

The other effect is the variable tension force stretching the rope due to the swinging of the pendulum. If we consider the amplitude of the pendulum $5\,^\circ$, the changes of the tension will be of the order of

$$\Delta F = mg(1-\cos\varphi) \approx 4\cdot 10^{-2}\,\mathrm{N}$$

To convert this to a change in length, we use the formula describing elasticity $\Delta l = k\Delta F$, where k represents the stiffness, which we calculate as $k = l_0/(ES)$, where S is the cross-section of the fiber and E is the modulus of elasticity. Assume a fibre thickness of $d = 0.1\,\mathrm{mm}$ a circular cross-section and a modulus of elasticity in tension of steel[8] $E = 220\,\mathrm{GPa}$. For the change in length, we then get

$$\Delta l = \frac{l_0}{ES}\Delta F \approx 10^{-3}\,\mathrm{m{\cdot}N^{-1}}\cdot 4\cdot 10^{-2}\,\mathrm{N} \approx 4\cdot 10^{-5}\,\mathrm{m}\,.$$

To achieve an accuracy of 10^{-6} we would need a maximal angle $0.1\,^\circ$, which would give a projection of the swing in the horizontal direction approximately $3\,\mathrm{mm}$.

[7] `http://kabinet.fyzika.net/studium/tabulky/tepelna-kapacita-roztaznost.`
[8] `http://kabinet.fyzika.net/studium/tabulky/modul-pruznosti.php`

Physical pendulum We will now look at our model of the physical pendulum, i.e., we will no longer consider our pendulum a point mass on a massless hinge but as a rigid sphere and a rigid rod (both from steel). The equation for the period of the physical pendulum is

$$T = 2\pi\sqrt{\frac{I}{mgl}}\,, \tag{1}$$

where l is the distance of the center of gravity from the turning point and I is the moment of inertia about the axis of rotation. Considering the density of steel $\rho = 7\,850\,\mathrm{kg{\cdot}m^{-3}}$, we calculate the mass of the fiber modeled by the rigid rod $m_\mathrm{f} = 0.12\,\mathrm{g}$ and the radius of the sphere $r = 3.12\,\mathrm{cm}$. The difference in the position of the center of gravity due to the fiber will be about 0.1 mm. We can express the moment of inertia as the sum of the moment of inertia of the rigid rod $I_\mathrm{rod} = m_\mathrm{f}(l-r)^2/3$ and the moment of inertia of the sphere $I_\mathrm{s} = 2m_\mathrm{s}r^2/5$, which must be shifted by the distance of the sphere's center from the center of gravity l using Steiner's theorem. We can now express the gravitational acceleration from the equation (1) and substitute for all the variables:

$$\begin{aligned}
g &= 4\pi^2\frac{I}{mlT^2} = \\
&= 4\pi^2\frac{\frac{2}{5}mr^2 + ml^2 + \frac{1}{3}m_\mathrm{f}(l-r)^2}{(m+m_\mathrm{f})\left(l - \frac{m_\mathrm{f}l}{2(m+m_\mathrm{f})}\right)T^2} = \\
&= 4\pi^2\frac{l}{T^2}\frac{\frac{2}{5}\frac{r^2}{l^2} + 1 + \frac{1}{3}\frac{m_\mathrm{f}}{m}\left(1-\frac{r}{l}\right)^2}{\left(1+\frac{m_\mathrm{f}}{m}\right)\left(1-\frac{m_\mathrm{f}}{2(m+m_\mathrm{f})}\right)} = \\
&= g_0\frac{\frac{2}{5}\frac{r^2}{l^2} + 1 + \frac{1}{3}\frac{m_\mathrm{f}}{m}\left(1-\frac{r}{l}\right)^2}{\left(1+\frac{m_\mathrm{f}}{m}\right)\left(1-\frac{m_\mathrm{f}}{2(m+m_\mathrm{f})}\right)} \approx \\
&\approx 1.000\,076 g_0
\end{aligned}$$

The difference in measured gravitational acceleration when we use the model of the physical pendulum is on the order of $7\cdot 10^{-4}\,\mathrm{m{\cdot}s^{-2}}$. The following question would be, how accurately the specific quantities involved in the calculation would have to be measured. We would again follow the propagation of uncertainty formula, but for the complexity of the calculation, we will not express it in this problem solution.

Influence of the air resistance As we determined in the previous section, the weight has a non-zero dimension, therefore a drag force acts on it as it moves through the air. In order to apply the theory of damped harmonic motion, we will consider that the drag force is directly proportional to the velocity of the body, so the flow is laminar, and we can write

$$F_\mathrm{d} = 6\pi\mu r v\,,$$

where r is the radius of the sphere and μ is the dynamic viscosity of the air, which has a value of about $\mu = 1.9\cdot 10^{-5}\,\mathrm{kg{\cdot}m^{-1}{\cdot}s^{-1}}$ and is very temperature dependent. The equation of motion now has the form

$$I\ddot{\varphi} + 6\pi\mu r l\dot{\varphi} + mgl\sin\varphi = 0\,,$$

which after linearization and dividing by the moment of inertia, we modify to

$$\ddot{\varphi} + \frac{6\pi\mu rl}{I}\dot{\varphi} + \frac{mgl}{I}\sin\varphi = 0\,,$$

that we identify as the equation for the damped oscillation $\ddot{x}+2\delta\dot{x}+\omega_0^2 x = 0$ with parameters $\delta = 3\pi\mu rl^2\dot{\varphi}/I$ and $\omega_0^2 = mgl/I$. Such a system will have its natural frequency ω modified to

$$\omega = \sqrt{\omega_0^2 - \delta^2}\,,$$

from which we can calculate the modified period

$$T = \frac{2\pi}{\sqrt{\left(\frac{2\pi}{T_0}\right)^2 - \delta^2}} = \frac{T_0}{\sqrt{1 - \frac{\delta^2 T_0^2}{4\pi^2}}} \approx 1.000\,001 T_0\,.$$

The air resistance gives a relative period change of 10^{-6}, hence the relative difference in the measured gravitational acceleration will be approximately twice as large $2\cdot 10^{-5}\,\mathrm{m\cdot s^{-2}}$. The amplitude then decreases as $e^{-\delta t}$, the characteristic decay constant (when the amplitude decreases to $1/e$) is approximately $1.7\cdot 10^5\,\mathrm{s} \doteq 50\,\mathrm{h}$. The decrease of amplitude will therefore be more affected by friction in the hinge than air resistance.

Other effects that could play a role here are the thermal expansion of the sphere or the influence of changes in temperature and pressure on the density and dynamic viscosity of the air. However, these again will not be explicitly calculated for their difficulty.

Influence of anharmonicity When solving the harmonic oscillator equation, we use approximation $\sin\varphi = \varphi$, which holds for small displacements. However, when we want to be more accurate, we can use the expansion of the sine to a higher order, so $\sin\varphi = \varphi + \varphi^3/6$, which gives us a solution for period[9]

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{4}\sin^2\frac{\varphi}{2}\right)\,.$$

Thus, the relative change in period for the maximum angle $5\,^\circ$ is approximately $5\cdot 10^{-4}$, so the difference of the gravitational acceleration compared to a simple calculation is of the order of $10^{-2}\,\mathrm{m\cdot s^{-2}}$. If we wanted an even more accurate result, we could use the development of the sine to an even higher order, but for these, we no longer have the exact analytical solution. However, due to the air resistance, the maximal angle generally decreases with time. Consequently, neither the solution in this part is absolutely accurate.

Other effects Another significant effect (see section Influence of air resistance) is the resistance in the hinge, which is difficult to quantify. However, experience tells us that it takes the pendulum less than 50 h to reduce its amplitude to a third of its original value. Thus, the effect of resistance is not insignificant. Another effect we would have to control to get exact results is the airflow in the room. It should ideally be non-existent, but that is technically difficult to achieve, so we should at least try to keep it stable. However, the stability of the flow can be disturbed by the movement of the experimenter or other people in the room.

[9] `https://phys.libretexts.org/Bookshelves/Classical_Mechanics/Classical_Mechanics_(Dourmashkin)/24%3A_Physical_Pendulums/24.04%3A_Appendix_24A_Higher-Order_Corrections_to_the_Period_for_Larger_Amplitudes_of_a_Simple_Pendulum`

Conclusion

In the first part of the solution, we have shown how big of an influence the individual changes in the gravitational acceleration have on the result. In the second part, we looked at all the considerations and corrections we would have to include to achieve the accuracy we were looking for. We see that from our model of $1\,\mathrm{kg}$ weight on a $2\,\mathrm{m}$ rod, we were unable to measure the length and time to greater accuracy than $10^{-6}\,\mathrm{m{\cdot}s^{-2}}$, which is an order of magnitude affected by the composition of the subsurface with the inclusion of the exact geographic location and altitude. Measuring the influence of cosmic bodies is completely beyond our capabilities. To achieve an accuracy of $10^{-6}\,\mathrm{m{\cdot}s^{-2}}$ we would need to measure the input parameters very accurately and monitor their changes. Apart from that, we would have to consider the fact that the physical pendulum is damped by air resistance, and finally, we would have to factor in the inaccuracy of the linearization of the problem. Measuring quantities very accurately is very difficult not only from a practical point of view of the need for careful measurement and accurate equipment, but the limits are set by the complexity of the theory used as well.

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