Problem V.4 ... Dark Side Time

8 points; průměr 6,94; řešilo 53 studentů

FYKOS plans to send its own satellite into space. It will be powered by solar cells; hence, it cannot stay in the Earth's shadow for too long. What is the height above the Earth's surface for which the time of the satellite passing through the Earth's shadow is the shortest? In your calculations, assume (same as the organizers did) that the Earth is a perfect sphere, that sunrays close to Earth's surface are parallel, and that the Sun, the Earth and the satellite's trajectory are in the same plane.
Bonus *While solving the problem, you will encounter an analytically unsolvable equation. Do not use online solvers, but try to create your own solution.*

Honza's batteries died in Kerbal.

First, we calculate the angular velocity of the satellite's orbit ω.

$$F_g = F_d \,,$$
$$G\frac{mM}{r^2} = \frac{mv^2}{r} \,,$$
$$G\frac{M}{r^2} = \omega^2 r \,,$$
$$\omega = \sqrt{\frac{GM}{r^3}} \,,$$

where M is the mass of the Earth, m is the satellite's mass, and r is the distance between the satellite and the Earth's center of mass. Next, we need to find out how much of the trajectory is spent in the shadow of the Earth. Hence, we calculate the angular size θ of the circle hidden in the Earth's shadow.

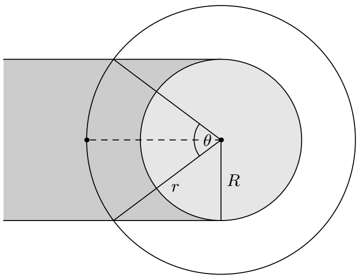

Fig. 1: Sketch of the situation. Here, R is Earth's radius.

The figure shows us that

$$\sin\left(\frac{\theta}{2}\right) = \frac{R}{r} \,,$$
$$\theta = 2\arcsin\left(\frac{R}{r}\right) \,,$$

where R is the radius of the Earth. Time spent in shadow t will then be

$$t = \frac{\theta}{\omega}\,,$$
$$t = 2\arcsin\left(\frac{R}{r}\right)\sqrt{\frac{r^3}{GM}}\,.$$

We want to minimize this time by a suitable choice of the parameter r. We derive the time with respect to r and set the resulting expression equal to zero.

$$\frac{\mathrm{d}t}{\mathrm{d}r} = 0\,,$$
$$\frac{\mathrm{d}}{\mathrm{d}r}\left(2\arcsin\left(\frac{R}{r}\right)\sqrt{\frac{r^3}{GM}}\right) = 0\,,$$
$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\frac{3}{2}}\arcsin\left(\frac{R}{r}\right)\right) = 0\,,$$
$$\frac{1}{\sqrt{1-\left(\frac{R}{r}\right)^2}}\frac{-R}{r^2}\left(\frac{r}{R}\right)^{\frac{3}{2}} + \arcsin\left(\frac{R}{r}\right)\frac{3}{2R}\left(\frac{r}{R}\right)^{\frac{1}{2}} = 0\,.$$

The equation we got looks very unpleasant. Nevertheless, we can use a trick to simplify it – we will make it dimensionless. That is easy as we have just one unknown and one parameter, and both have a length dimension. We can rephrase the problem by removing one parameter and getting a new dimensionless equation. We introduce a normalized distance u, plug it in, and modify the expression.

$$u = \frac{R}{r}\,,$$
$$\frac{3}{2}\arcsin(u) - \frac{1}{\sqrt{u^{-2}-1}} = 0\,.$$

We cannot solve this equation analytically. However, we can use programs such as Desmos or Wolfram Alpha. We get the result

$$u \doteq 0.823\,.$$

After reestablishing initial quantities, we get

$$r \doteq \frac{R}{0.823}\,,$$
$$r \doteq 1.215R\,.$$

Since r is the orbital radius measured from the Earth's center, we subtract R from both sides of the equation. That gives the height above the surface h on the left side and on the other side a value in multiples of R

$$h \doteq 0.215R\,.$$

After inserting $R \doteq 6\,378\,\mathrm{km}$ we get the final result

$$h \doteq 1\,370\,\mathrm{km}\,.$$

So far, we established a local extreme of the function. However, we would like to know if it is a global minimum as well. Let's compute the second derivative of $t(r)$.

$$\frac{\mathrm{d}^2 t}{\mathrm{d}r^2} = \frac{2}{\sqrt{GMr}}\left(\frac{3}{4}\arcsin\left(\frac{R}{r}\right) + \frac{-\frac{r^2}{R^2}+2}{\sqrt{\frac{r}{R}^2-1}^3}\right)\,,$$

$$\frac{\mathrm{d}^2 t}{\mathrm{d}r^2}(1.215R) \doteq \frac{4.207}{\sqrt{GMR}}\,.$$

Since it is positive, the function is convex, and we have really found a local minimum.

Even though we do not have other candidates for the extreme, we still have to investigate the endpoints of the function's domain. In point R, the function gains value $pi^{sqrt}\frac{R^3}{GM}$, which is higher than the functional value at the point $2.589\sqrt{\frac{R^3}{GM}}$ we found.

Since $\arcsin(x) \approx x$ for $x \approx 0$, the function $t(r)$ in infinity behaves as

$$t(r) = 2\arcsin\left(\frac{R}{r}\right)\sqrt{\frac{r^3}{GM}} \approx 2\frac{R}{r}\sqrt{\frac{r^3}{GM}} = 2R\sqrt{\frac{r}{GM}}$$

and in limit to infinity, grows beyond all limits. We have indeed found the global minimum. Of course, instead of this calculation, it was possible to have the function drawn by some program.

Bonus

We modify the dimensionless equation into the form

$$\frac{3}{2}\arcsin(u) - \frac{1}{\sqrt{u^{-2}-1}} = 0\,.$$

If we have an equation of the form

$$f(x) = 0\,,$$

we can (under certain assumptions) find its roots using the so-called Newton method, which works as follows. First, we try to estimate the value of x_0. This estimate will be further refined by using the recurrent formula

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}\,,$$

which, in our case looks like

$$u_{k+1} = u_k - \frac{\frac{3}{2}\arcsin(u_k) - \frac{1}{\sqrt{u_k^{-2}-1}}}{\frac{3}{2}\frac{1}{\sqrt{1-u_k^2}} - \frac{u_k^{-3}}{\sqrt{u_k^{-2}-1}^3}}\,,$$

$$u_{k+1} = \frac{3 - 5u_k^3 - 3\sqrt{1-u_k^2}^3\arcsin(u_k)}{1-3u_k^2}\,.$$

If our initial estimate was a good one[1], the new elements of the obtained sequence will get closer to the value of the solution of the original equation. For initial condition $u_0 = 0.9$ we get the sequence

Tab. 1: Result values in each iteration.

k	$\frac{u_k}{1}$
0	0.900 0
1	0.829 4
2	0.823 6
3	0.823 4
4	0.823 4

From the table 1 we see that already in the 3rd iteration we got to the required precision.

A few notes at the end

If we loosen up on our assumptions, we get two more solutions.

The first would be a heliosynchronous orbit. The imperfectly spherical shape of the Earth causes the orbits[2] of satellites to rotate slowly, which we can exploit. If our satellite's orbit has the right parameters[3], it will have a period of one year. Thus, it can be ensured that the orbital plane of the satellite has a constant orientation with respect to the Earth-Sun line. If we choose such an orbit that does not enter the shadow of the Earth during the first orbit, it will not enter the shadow of the Earth in the long term.

The second solution would be to put the satellite in one of the Lagrange points. However, this solution is somewhat problematic. The first three are not stable, yet they are (the first two of them) used by many satellites like the SOHO satellites observing the Sun at point L1 or the James Webb telescope at point L2. Point L3 is on the opposite side of the Sun from the Earth, and a potential satellite near this point would be difficult to communicate with. Points L4 and L5 are 1 au away from Earth, so a satellite would probably be too far away to serve its purpose.

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[1] in our case, the initial value must be in the interval $(0.696, 1)$. Otherwise, Newton's method will not converge to a given root, but to 0.

[2] technically called Axial precession

[3] Here, the inclination of the plane in which the satellite orbits is important with respect to the plane of the Earth's equator. For those interested, details, for example, at https://en.wikipedia.org/wiki/Sun-synchronous_orbit.