Problem II.5 ... ferry

10 points; průměr 4,83; řešilo 63 studentů

Imagine a ferry in the shape of a rectangular cuboid with a weight M, length L, width W, and height $H \ll L$ from the keel to the deck. After docking at the pier, passengers gradually exit through the back of the deck so that the empty front part of the deck becomes larger and the area density of people on the filled part does not change in a different way. Find the maximum weight of passengers the ferry can carry so that no part of the deck is below the surface when people disembark. Consider that the ship is stable in the transverse direction and that people get off slowly.

After quite some time, Dodo was at sea again.

Suppose people disembark from the boat slowly enough (compared to the boat's oscillations). In that case, we can determine the boat's position at any moment – its depth of immersion and tilt – from the balance of forces acting on the boat and their moments. The boat is subject to buoyant force at the center of gravity of the submerged part of the boat, the gravitational force of the boat at its center of gravity (which, in the case of a cuboid, is in its geometric center), and the gravitational force of the people still on board. The disembarkation of people will gradually reduce the overall downward force, thus decreasing the depth of immersion and shifting the location of the gravitational force of the people to the side, causing the boat to tilt. Let's denote the length of the deck filled with people as $l \leq L$, the mass of passengers on the boat as m, the tilt of the deck relative to the water surface as θ, and the depth of immersion as ξ – the length of the submerged part of the line segment leading from the center of the deck to the center of the keel.

The relation describing the force equilibrium is

$$\rho g V_\mathrm{p} = Mg + m\frac{l}{L}g \quad \Rightarrow \quad \rho V_\mathrm{p} = M + m\frac{l}{L}\,,$$

where ρ is the density of water. The volume of the immersed part of the boat in the shape of a rectangular cuboid is defined as the product of its width and side cut of the immersed part shaped as a trapezoid. We can easily confirm that $V_\mathrm{p} = DL\xi$ holds, and for depth of immersion, we get

$$\xi = \frac{M + m\frac{l}{L}}{LD\rho}\,.$$

This relation implies the condition $\xi \leq H$ which determines the maximum possible useful ship's carrying capacity when the cargo is uniformly distributed on the ship $l = L$

$$m_\text{max, static} = LDH\rho - M\,.$$

Determining the balance of the torques will be more difficult because of the geometry. Consider the torques around the bottom edge of the ship on the side that people are exiting. Including the contributions of the gravitational and buoyant forces, we have

$$\left(\frac{L}{2}\cos\theta - \frac{H}{2}\sin\theta\right)Mg + \left(\frac{l}{2}\cos\theta - H\sin\theta\right)m\frac{l}{L}g - \left(x_\mathrm{T}\cos\theta - y_\mathrm{T}\sin\theta\right)\rho g L D\xi = 0\,,$$

where x_T and y_T are the distance of the center of the submerged part of the ship from the reference corner in the diameter of the length and height of the ship. From this, by further substituting for the ξ we obtain

$$\frac{ML}{2} + \frac{ml^2}{2L} - \left(M + m\frac{l}{L}\right)x_\mathrm{T} = \tan\theta\left(M\frac{H}{2} + m\frac{l}{L}H - y_\mathrm{T}\left(M + m\frac{l}{L}\right)\right)\,.$$

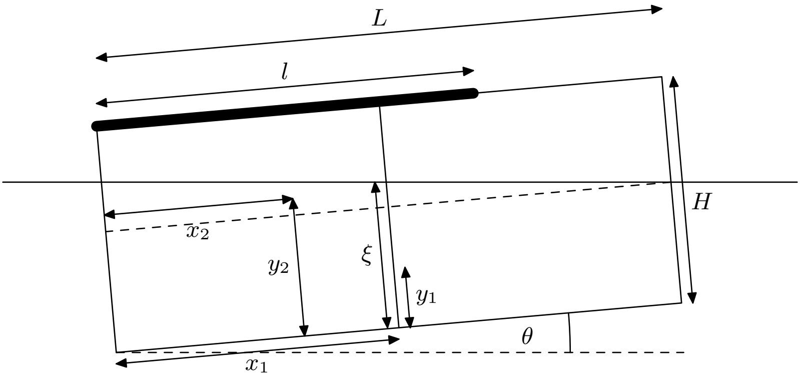

Fig. 1: Sketch of the situation from the side and with the positions of the centers of gravity of the individual submerged parts.

The situation would be simpler if x_T and y_T did not depend on θ, which is false. To calculate the position of the center of gravity, we take the submerged part of the longitudinal cross-section of the trajectory; let us divide the rectangle lying on the bottom of the ship which reaches up to the lowest submerged point on the edge opposite the exit and the remaining triangle. For determining x_T, we know that a rectangle has a center of gravity in $x_1 = L/2$ and a triangle in $x_2 = L/3$ (the triangle's center of gravity divides the line of gravity by a third). For y_T we have the center of gravity of the rectangle in $y_1 = (\xi - L/2\tan\theta)/2$[1] and of a triangle $y_2 = \xi - L/2\tan\theta + L/3\tan\theta$. Further, the masses of the bodies are proportional to their areas $S_1 = 2Ly_1$, $S_2 = 1/2L^2\tan\theta$. Thus, for the position of the center of the submerged part of the ship, we have

$$x_\mathrm{T} = \frac{\frac{L}{2}L2y_1 + \frac{L}{3}\frac{L^2}{2}\tan\theta}{\xi L} = \frac{L}{2}\left(1 - \frac{1}{6}\frac{L}{\xi}\tan\theta\right)\,,$$

$$y_\mathrm{T} = \frac{\left(\xi - \frac{L}{2}\tan\theta\right)^2\frac{L}{2} + \left(\xi - \frac{L}{6}\tan\theta\right)\frac{L^2}{2}\tan\theta}{\xi L} = \frac{\xi}{2}\left(1 + \frac{1}{12}\left(\frac{L}{\xi}\right)^2\tan^2\theta\right)\,.$$

Substituting into the previous relation for the equilibrium of moments, we get

$$\frac{lm}{2}\frac{l-L}{L} + \frac{L^3D\rho}{12}\tan\theta = \tan\theta\left[M\frac{H}{2} + m\frac{l}{L}H - \frac{\left(M + m\frac{l}{L}\right)^2}{2LD\rho}\left(1 + \frac{1}{12}\left(\frac{L}{\xi}\right)^2\tan^2\theta\right)\right]\,.$$

We have obtained the cubic equation for $\tan\theta$, whose analytical calculation is impractical. However, the situation simplifies in the given approximation $H \ll L$. Therefore, we need to be careful not to lose essential terms. The first term on the left side is the absolute term, which we will keep for now. We will move the second term on the left side to the right. The first two terms on the right side are about MH, comparable to $\rho LDHH$ (the mass of water displaced

[1] At this point, it would be useful to draw attention to the fact that the whole calculation assumes that this point does not arise from water, i.e., that $2\xi > L\tan\theta$.

by the boat is comparable to the mass when it is not submerged). However, due to the given condition, we know that $\rho LDHH \ll \rho L^3 D$, so we can neglect this term. The term with the last parenthesis is still uncertain – the expression $(L/\xi)^2 \tan^2\theta$ is a product of a large and a small number, but we expect that if the other end of the boat does not emerge, its value should be less than 4. However, this statement will need to be verified, not assumed. For now, after a slight modification by multiplication and substitution for ξ, we have

$$\frac{lm}{2}\frac{L-l}{L} \approx \tan\theta \left[\frac{L^3 D\rho}{12} + \frac{\left(M + m\frac{l}{L}\right)^2}{2LD\rho} + \frac{LD\rho}{2}\frac{1}{12}L^2\tan^2\theta\right] .$$

From this form of the relation, we see that the term with tangent on the right-hand side is the same as the first term of the right-hand side. Since $2 \gg \tan^2\theta$, we can ignore this term, which turns the cubic equation into a linear equation. Next, for the middle term of the left-hand side, we have order-of-magnitude estimates

$$\frac{\left(M + m\frac{l}{L}\right)^2}{2LD\rho} \propto \frac{(LDH\rho)^2}{LD\rho} = LD\rho H^2 \ll \frac{L^3 D\rho}{12} ,$$

therefore, we can neglect this term for a sufficiently long boat. The final relation for the angle of the heel is thus[2]

$$\tan\theta \approx \frac{lm}{2}\left(1 - \frac{l}{L}\right)\frac{12}{L^3 D\rho} = \frac{6ml\,(L-l)}{L^4 D\rho} .$$

It remains, then, to fit this inclination and submergence to the condition of the immersion of the deck at the point of ascent

$$H > \xi + \frac{L}{2}\tan\theta = \frac{M + m\frac{l}{L}}{LD\rho} + \frac{l}{L}\frac{L-l}{L}\frac{3m}{LD\rho} ,$$

$$LD\rho H - M > m\frac{l}{L} + 3m\frac{l}{L}\left(1 - \frac{l}{L}\right) = m\frac{l}{L}\left(4 - 3\frac{l}{L}\right) .$$

The right-hand side is quadratic in $l/L = x$, so its maximum is midway between the roots of the function $f(x) = x(4-3x)$, so for $x_c = l_c/L = 2/3$. Thus, we must satisfy the condition

$$LD\rho H - M > \frac{4m}{3} ,$$

[2] Using this relation, let us verify the assumption in the previous remark.

$$L\tan\theta = \frac{l}{L}\frac{L-l}{L}\frac{6m}{LD\rho} < 2\frac{M + m\frac{l}{L}}{LD\rho} = 2\xi$$

is equivalent to a condition

$$\frac{l}{L}\left(1 - \frac{l}{L}\right)3m < M + m\frac{l}{L} ,$$

$$\left(2 - 3\frac{l}{L}\right)\frac{l}{L}m < M ,$$

where the left-hand side has maximums on the interval $0 < l < L$ at point $l = L/3$ with value

$$\left(2 - 3\frac{l}{L}\right)\frac{l}{L}m \le \frac{m}{3} < M .$$

so for the weight of passengers we obtain

$$m_{\text{max, dynamic}} < \frac{3}{4}\left(LD\rho H - M\right) = \frac{3}{4} m_{\text{max, static}}$$

which is three-quarters of the maximum load during the voyage.[3]

The ferry may carry passengers weighing not more than three-quarters of the payload displacement if they are disembarked from the boat by successive ejection in the back, as described above. The ship's edge from which ascent is made shall be nearest to the water surface after one-third of the passengers have disembarked. Finally, let us add that the situation will differ for a realistic ship since we have considered $L \gg H$, even more than $L > 100H$.

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[3] Let's finish our control consideration. Substituting the worst case $m = 3M$ we get

$$LD\rho H > 5M = 5LDH\rho_{\text{L}} \,,$$

thus for the density of the ship $\rho_{\text{L}} < \rho/5 = 200\,\mathrm{kg/m^3}$, which also means that the draft of such a ship, when empty, would be only $1/5$. In reality, the draft of an empty ship is usually somewhat higher. In some cases – large tankers – it can still be half as much. So let's settle it: We've solved the problem for most ships. Otherwise, we would have had to calculate again for the case where the submerged part is triangular. If to be realistic, real ships, moreover, are not prism-shaped, and calculation would therefore have to be done numerically.