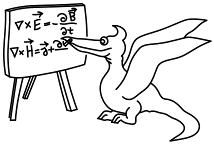

Serial: Polarisation

In the previous episode of the serial, we explored the physics of waves in one dimension – the string could oscillate only in the vertical direction, the particle was represented by a single component wavefunction etc. Now, it is time to describe waves in more than one dimension, which are interconnected. For example, we could think about string oscillating in both directions perpendicular to the equilibrium string direction, or think about waves in a charged liquid, where local mass density, temperature and even charge density can oscillate. As specific examples, we will consider slow waves in a plasma, which arise from the equations of magnetohydrodynamics. For now though, let's start with a more elementary example of a string oscillating in two perpendicular directions.

Jump Rope

Consider a jump rope taut between two points, so that the tension in the jump rope is T. The length mass density of the jump rope is ρ. Let the system of coordinates have an origin at one of the points where the jump rope ends, and let the x axis be parallel to the direction of the equilibrium jump rope shape. We will denote u the displacement of the jump rope from the equilibrium position in the vertical direction (along the z axis) and v the displacement of the jump rope in the horizontal direction (along y axis, perpendicularly to the tension direction). A similar analysis can be carried out for each component separately, but it a more insightful strategy is to redo the derivation in vector formalism.

We have derived that the force acting on a string element of length $\mathrm{d}x$ in the horizontal direction is $\mathrm{d}F = T\frac{\partial^2 u}{\partial x^2}\,\mathrm{d}x$. It is reasonable to assume that in this case, the oscillations in the vertical direction will be independent of the oscillations in the horizontal direction, and hence we can write for the force in the vertical direction $\mathrm{d}F' = T\frac{\partial^2 v}{\partial x^2}\,\mathrm{d}x$, and therefore the total force vector satisfies

$$\mathrm{d}\mathbf{F} = T\frac{\partial^2 \mathbf{u}}{\partial x^2}\,\mathrm{d}x\,,$$

where $\mathbf{u} = (u, v)^\mathsf{T}$ and $\mathrm{d}\mathbf{F} = (\mathrm{d}F, \mathrm{d}F')^\mathsf{T}$. Newton's second law can be then written in vector form as

$$\mathrm{d}\mathbf{F} = \mathrm{d}m\frac{\partial^2 \mathbf{u}}{\partial t^2} = \rho\,\mathrm{d}x\frac{\partial^2 \mathbf{u}}{\partial x^2}$$

and therefore

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 \mathbf{u}}{\partial x^2}\,.$$

This is the two dimensional variant of the wave equation. Importantly, both displacements are still functions of one time variable and one position variable, and hence we can do the Fourier substitution as we are used to, which leads to

$$-\omega^2\hat{\mathbf{u}} = \frac{T}{\rho}\left(-k^2\right)\hat{\mathbf{u}}\,,$$

where $\hat{\mathbf{u}}$ is the complex vector displacement, and we have $\mathbf{u} = \operatorname{Re}\hat{\mathbf{u}}$, where the real part is taken for each component of the vector separately. This can be seen as a set of algebraic equations, which can be written in the matrix form as

$$\begin{pmatrix} \frac{T}{\rho}k^2 - \omega^2 & 0 \\ 0 & \frac{T}{\rho}k^2 - \omega^2 \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ,$$

where $\hat{u}$ and $\hat{v}$ are the complex displacements in the separate directions. The Fourier substitution assumes forms of the solution

$$\hat{u} = u_0 \mathrm{e}^{\mathrm{i}k\ x - \mathrm{i}\omega t} ,$$
$$\hat{v} = v_0 \mathrm{e}^{\mathrm{i}k\ x - \mathrm{i}\omega t} ,$$

where u_0 and v_0 are potentially complex constants. Since the exponential part is identical for both directions, the matrix equation can be rewritten as

$$\begin{pmatrix} \frac{T}{\rho}k^2 - \omega^2 & 0 \\ 0 & \frac{T}{\rho}k^2 - \omega^2 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

To proceed, we recall the formalism used for the description of normal modes – a matrix equation of this type has a non-trivial solution only if the determinant of the matrix is zero, and therefore

$$0 = \begin{vmatrix} \frac{T}{\rho}k^2 - \omega^2 & 0 \\ 0 & \frac{T}{\rho}k^2 - \omega^2 \end{vmatrix} = \left(\frac{T}{\rho}k^2 - \omega^2\right)^2 \quad \Rightarrow \quad \omega = \pm\sqrt{\frac{T}{\rho}}k \ .$$

We can see that we have recovered the dispersion relation, identical to the 1D one. The matrix that satisfies this condition becomes a zero matrix, and therefore there are no conditions on u_0 and v_0, i.e. the wave on the string can be written as an arbitrary linear combination

$$\hat{\mathbf{u}}(x,t) = u_0 \mathrm{e}^{\pm\mathrm{i}k\ x - \mathrm{i}\omega t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + v_0 \mathrm{e}^{\pm\mathrm{i}k\ x - \mathrm{i}\omega t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} ,$$

where the directions of propagation of the wave can be independent. The two component waves of the overall wave are called the polarisations of the wave and the vector $(u_0, v_0)^\mathsf{T}$ is called the polarisation vector. How can we determine the constants u_0 and v_0? We need to consider a specific wave to do that. So, consider a wave that causes the jump rope to move as when being jumped over, i.e. it circles around the equilibrium position so that each element of the rope moves along a circle with constant angular velocity, with radii of the circles differing for each element. The radii increase towards the centre of the jump rope, where the amplitude reaches the maximum. At time $t = 0$, the jump rope vector displacement can be written as

$$\mathbf{u}(x,0) = \begin{pmatrix} A\ \sin\left(\frac{\pi x}{L}\right) \\ 0 \end{pmatrix} ,$$

where A is a real constant. We also need to know the speed of the jump rope at time $t = 0$, which can be written as

$$\frac{\partial \mathbf{u}}{\partial t} = \begin{pmatrix} 0 \\ A\ \omega \sin\left(\frac{\pi x}{L}\right) \end{pmatrix} .$$

How can this motion be described in terms of wave polarisations? For a standing wave, we would expect the motion to be a superposition of waves moving in opposite directions. Hence, we propose the following form of the solution

$$\hat{\mathbf{u}}(x,t) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \mathrm{e}^{\mathrm{i}kx - \mathrm{i}\omega t} + \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \mathrm{e}^{-\mathrm{i}k\ x - \mathrm{i}\omega t} .$$

At time $t = 0$ we therefore have

$$\hat{\mathbf{u}}(x,0) = \begin{pmatrix} u_0 \mathrm{e}^{\mathrm{i}k\ x} + u_1 \mathrm{e}^{-\mathrm{i}k\ x} \\ v_0 \mathrm{e}^{\mathrm{i}k\ x} + v_1 \mathrm{e}^{-\mathrm{i}k\ x} \end{pmatrix} = \begin{pmatrix} (u_0 + u_1)\cos(kx) + \mathrm{i}\,(u_0 - u_1)\sin(kx) \\ (v_0 + v_1)\cos(kx) + \mathrm{i}\,(v_0 - v_1)\sin(kx) \end{pmatrix} .$$

The real part can be determined as

$$\mathbf{u}(x,0) = \begin{pmatrix} (\operatorname{Re} u_0 + \operatorname{Re} u_1)\cos(kx) + (\operatorname{Re}(\mathrm{i}u_0) - \operatorname{Re}(\mathrm{i}u_1))\sin(kx) \\ (\operatorname{Re} v_0 + \operatorname{Re} v_1)\cos(kx) + (\operatorname{Re}(\mathrm{i}v_0) - \operatorname{Re}(\mathrm{i}v_1))\sin(kx) \end{pmatrix} .$$

In order to satisfy the initial conditions, we require $k = \frac{\pi}{L}$ and furthermore

$$\begin{aligned} \operatorname{Re} u_0 + \operatorname{Re} u_1 &= 0 , \\ \operatorname{Re}(\mathrm{i}u_0) - \operatorname{Re}(\mathrm{i}u_1) &= A , \\ \operatorname{Re} v_0 + \operatorname{Re} v_1 &= 0 , \\ \operatorname{Re}(\mathrm{i}v_0) - \operatorname{Re}(\mathrm{i}v_1) &= 0 . \end{aligned}$$

Since for any complex number

$$\operatorname{Re}(\mathrm{i}z) = -\operatorname{Im} z$$

applies, we can write

$$\begin{aligned} \operatorname{Re} u_0 &= -\operatorname{Re} u_1 , \\ \operatorname{Re} v_0 &= -\operatorname{Re} v_1 , \\ \operatorname{Im} v_0 &= \operatorname{Im} v_1 , \\ \operatorname{Im} u_0 &= \operatorname{Im} u_1 - A . \end{aligned}$$

These are the first four equations for a total of eight unknowns (real and imaginary parts of $u_{0,1}$ and $v_{0,1}$). The other four equations can be derived from the equation for initial velocity of the jump rope. Specifically,

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} = (-\mathrm{i}\omega)\,\hat{\mathbf{u}}$$

and therefore

$$\frac{\partial \mathbf{u}}{\partial t} = \omega \begin{pmatrix} (\operatorname{Re}(-\mathrm{i}u_0) + \operatorname{Re}(-\mathrm{i}u_1))\cos(kx) + (\operatorname{Re} u_0 - \operatorname{Re} u_1)\sin(kx) \\ (\operatorname{Re}(-\mathrm{i}v_0) + \operatorname{Re}(-\mathrm{i}v_1))\cos(kx) + (\operatorname{Re} v_0 - \operatorname{Re} v_1)\sin(kx) \end{pmatrix} .$$

So, the remaining four equations are

$$\begin{aligned} \operatorname{Im} u_0 &= -\operatorname{Im} u_1 , \\ \operatorname{Re} u_0 &= \operatorname{Re} u_1 , \\ \operatorname{Im} v_0 &= -\operatorname{Im} v_1 , \\ \operatorname{Re} v_0 &= \operatorname{Re} v_1 + A . \end{aligned}$$

The set is then solved by

$$\begin{aligned} u_0 &= -\mathrm{i}\frac{A}{2}\,, \quad v_0 = \frac{A}{2}\,, \\ u_1 &= \mathrm{i}\frac{A}{2}\,, \quad v_1 = -\frac{A}{2}\,. \end{aligned}$$

The complete time evolution of the wave is described by the expression

$$\hat{\mathbf{u}}(x,t) = \begin{pmatrix} -\mathrm{i}\frac{A}{2}\mathrm{e}^{\mathrm{i}k\ x} + \mathrm{i}\frac{A}{2}\mathrm{e}^{-\mathrm{i}k\ x} \\ \frac{A}{2}\mathrm{e}^{\mathrm{i}k\ x} - \frac{A}{2}\mathrm{e}^{-\mathrm{i}k\ x} \end{pmatrix} \mathrm{e}^{-\mathrm{i}\omega t} = A \begin{pmatrix} \sin\left(\frac{\pi}{L}x\right) \\ \mathrm{i}\sin\left(\frac{\pi}{L}x\right) \end{pmatrix} \mathrm{e}^{-\mathrm{i}\omega t}\,,$$

and for the real displacement

$$\mathbf{u}(x,t) = A\ \sin\left(\frac{\pi}{L}x\right) \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix}\,.$$

Waves in Plasma

The direction in which the oscillation of waves occur does not, however, have to be a spatial dimension. Instead, the oscillations can occur in different degree of freedom. This fact can be demonstrated on a model of plasma. Plasma consists of charged particles, nuclei and electrons. We will limit ourselves to slow dynamics, i.e. we will assume that any electron dynamic has equilibrated and balanced the electric field in the plasma. Furthermore, we will assume that values of all variables are only changing in the x direction of the cartesian system of coordinates (imagine a narrow column of plasma). In this case the equations of magnetohydrodynamics are the equations we need to solve. The equations are a set of two scalar differential equations and two vector differential equations. We will present them now.

First equation is the so called continuity equation, which ensures that the mass in plasma is conserved. The form of the equation is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho v_x\right) = 0\,,$$

where ρ is the mass density of plasma and $\mathbf{v} = (v_x, v_y, v_z)^\mathsf{T}$ is the velocity of the plasma. The other scalar equation is the state equation of the plasma. Generally, the specific form of this equation is hard to determine, so we use only a phenomenological variant, which states that

$$\left(\frac{\partial}{\partial t} + v_x\frac{\partial}{\partial x}\right)\left(\frac{P}{\rho^\gamma}\right) = 0\,,$$

where P is the pressure in the plasma and γ is a real constant. If we modelled plasma as an ideal gas, then γ would be the Poisson constant. The state equation ensures that the plasma compresses adiabatically.

We now proceed to the vector differential equations. First equation is the so called Navier-Stokes equation, which represents Newton's second law for fluids

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + v_x\frac{\partial}{\partial x}\mathbf{v}\right) = \begin{pmatrix} -\frac{\partial P}{\partial x} \\ 0 \\ 0 \end{pmatrix} - \frac{1}{\mu_0}\mathbf{B}\times\begin{pmatrix} 0 \\ -\frac{\partial B_z}{\partial x} \\ \frac{\partial B_y}{\partial x} \end{pmatrix}\,,$$

where $\mathbf{B} = (B_x, B_y, B_z)^\mathsf{T}$ is the magnetic field in plasma and $\times$ stands for a vector product. This equation is harder to grasp, but it can be understood as a balance of two terms. On the right-hand side, we have the forces acting on a unit volume of the fluid, both due to pressure imbalance and magnetic field. On the left-hand side, we have the change of momentum of the fluid per unit volume.

The last vector equation is the so called induction equation, which follows from Maxwell equations and in our model it has the following form

$$\frac{\partial \mathbf{B}}{\partial t} = \begin{pmatrix} 0 \\ -\frac{\partial (\mathbf{v}\times\mathbf{B})_z}{\partial x} \\ \frac{\partial (\mathbf{v}\times\mathbf{B})_y}{\partial x} \end{pmatrix} .$$

A direct solution of these equations is clearly an extremely difficult task – we have a set of non-linear differential equations. However, there exists a trivial solution describing the equilibrium state $\rho = \rho_0$, $P = P_0$, $\mathbf{v} = 0$ a $\mathbf{B} = \mathbf{B}_0$, where variables with index 0 are constants everywhere in space and time. The equations can be linearized around this equilibrium state, which leads to a wave equation.

Without the loss of generality, we can assume that $\mathbf{B}_0$ lies in the xz plane, i.e. $B_{y0} = 0$. In order to linearize the equations, we will assume that the form of variables is as $\rho = \rho_0 + \rho_1(x,t)$, where $|\rho_1| \ll |\rho_0|$ for all times and positions. To carry out the linearisation, we substitute these expressions into our equations and retain only terms up to first order in the "small" variables. Also, we use $\mathbf{v}_0 = \mathbf{0}$

First equation is linearised as

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial v_{1x}}{\partial x} = 0 .$$

Second equation requires a more detailed modification

$$\left(\frac{\partial}{\partial t} + v_{1x}\frac{\partial}{\partial x}\right)\left(\frac{P_0 + P_1}{(\rho_0+\rho_1)^\gamma}\right) = 0 .$$

Then, we can write

$$(\rho_0+\rho_1)^{-\gamma} = \rho_0^{-\gamma}\left(1 + \frac{\rho_1}{\rho_0}\right)^{-\gamma} \approx \rho_0^{-\gamma}\left(1 - \frac{\gamma\rho_1}{\rho_0}\right) .$$

Retaining only terms up to the first order, we have

$$\rho_0^{-\gamma}\left(\frac{\partial}{\partial t} + v_x\frac{\partial}{\partial x}\right)\left(P_0 - P_0\frac{\gamma\rho_1}{\rho_0} + P_1\right) = 0 .$$

Since P_0 is constant and other terms in the second bracket are first order, only the time derivative remains to the first order in our approximation, and hence (for non-zero density ρ_0)

$$\frac{\partial P_1}{\partial t} - \frac{\gamma P_0}{\rho_0}\frac{\partial \rho_1}{\partial t} = 0 .$$

Navier-Stokes equation needs to be separated into components. Also, lets define α as the angle between $\mathbf{B}_0$ and the x axis, so that $B_{0x} = B_0 \cos\alpha$, $B_{0z} = B_0 \sin\alpha$, where $B_0 = |\mathbf{B}_0|$.

Hence we can write (here without derivation, but the derivation does not contain any noteworthy tricks)

$$\begin{aligned}
\rho_0 \frac{\partial v_{1x}}{\partial t} &= -\frac{\partial P_1}{\partial x} - \frac{1}{\mu_0} B_0 \sin\alpha \frac{\partial B_{1z}}{\partial x}\,,\\
\rho_0 \frac{\partial v_{1z}}{\partial t} &= \frac{1}{\mu_0} B_0 \cos\alpha \frac{\partial B_{1z}}{\partial x}\,,\\
\rho_0 \frac{\partial v_{1y}}{\partial t} &= \frac{1}{\mu_0} B_0 \cos\alpha \frac{\partial B_{1y}}{\partial x}\,.
\end{aligned}$$

Similar decomposition can be used to determine three equations following from the induction equation

$$\begin{aligned}
\frac{\partial B_{1x}}{\partial t} &= 0\,,\\
\frac{\partial B_{1z}}{\partial t} &= B_0 \cos\alpha \frac{\partial v_{1z}}{\partial x} - B_0 \sin\alpha \frac{\partial v_{1x}}{\partial x}\,,\\
\frac{\partial B_{1y}}{\partial t} &= B_0 \cos\alpha \frac{\partial v_{1y}}{\partial x}\,.
\end{aligned}$$

In total, we have eight equations for eight unknowns – 3 components of $\mathbf{B}_1$, 3 components of $\mathbf{v}_1$ and ρ_1 and P_1.

In vector equations, we deliberately put the y components as last. The reason behind this is to highlight the fact that these equations are independent from the rest of the equations – unknowns v_{1y} and B_{1y} are present only in these two equations. This means that waves described by these two equations is independent of the waves described by the remaining five equations.

The next step is already standard – we carry out the Fourier substitution for all variables of type $\rho_1(x,t) \rightarrow \hat{\rho}_1 = A\mathrm{e}^{\mathrm{i}kx - \mathrm{i}\omega t}$, where A is a complex constant and $\hat{\rho}_1$ is the complex displacement (in this case mass density displacement). All eight equations can then be written as two matrix equations (one for unknowns v_{1y} and B_{1y}, one for the rest)

$$\begin{pmatrix} -\mathrm{i}\omega\rho_0 & -\mathrm{i}k\,\frac{1}{\mu_0} B_0\cos\alpha \\ \mathrm{i}k\;B_0\cos\alpha & \mathrm{i}\omega \end{pmatrix} \begin{pmatrix} \hat{v}_{1y} \\ \hat{B}_{1y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\,,$$

$$\begin{pmatrix} -\mathrm{i}\omega\rho_0 & 0 & \frac{\mathrm{i}k}{\mu_0} B_0 \sin\alpha & 0 & \mathrm{i}k \\ 0 & -\mathrm{i}\omega\rho_0 & -\frac{\mathrm{i}k}{\mu_0} B_0\cos\alpha & 0 & 0 \\ \mathrm{i}k\;B_0\sin\alpha & -\mathrm{i}k\;B_0\cos\alpha & -\mathrm{i}\omega & 0 & 0 \\ \mathrm{i}k\;\rho_0 & 0 & 0 & -\mathrm{i}\omega & 0 \\ 0 & 0 & 0 & \mathrm{i}\omega\frac{\gamma P_0}{\rho_0} & -\mathrm{i}\omega \end{pmatrix} \begin{pmatrix} \hat{v}_{1x} \\ \hat{v}_{1z} \\ \hat{B}_{1z} \\ \hat{\rho}_1 \\ \hat{P}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\,.$$

The trivial equation $-\mathrm{i}\omega\hat{B}_{1x} = 0$ is not included here, and it simply requires that $\hat{B}_{1x} = 0$. So, we have two potential wave types. In order to determine the dispersion relation, we need to find the determinant of the matrix of the corresponding wave. In the following analysis, we will only consider waves in v_{1y} and B_{1y}. These waves are called Alvén waves.

The determinant of the 2×2 matrix is readily determined as

$$\begin{vmatrix} \mathrm{i}kB_0\cos\alpha & \mathrm{i}\omega \\ -\mathrm{i}\omega\rho_0 & -\mathrm{i}k\frac{1}{\mu_0}B_0\cos\alpha \end{vmatrix} = k^2\frac{1}{\mu_0}B_0^2\cos^2\alpha - \omega^2\rho_0\,,$$

and from the condition that the determinant has to be equal to zero, we obtain the dispersion relation

$$\omega = \frac{B_0 \cos \alpha}{\sqrt{\mu_0 \rho_0}} k \ .$$

This means that the waves have linear dispersion (same as sound or light waves) with phase velocity

$$v_\mathrm{p} = \frac{B_0 \cos \alpha}{\sqrt{\mu_0 \rho_0}} \ .$$

So, for field $\mathbf{B}_0$ pointing further away from the x direction, the waves move slower. In order to determine the polarisation vectors of Alfvén waves, we substitute the result back into the matrix equation

$$\begin{pmatrix} \mathrm{i}k \ B_0 \cos \alpha & \mathrm{i}\frac{k \ B_0 \cos \alpha}{\sqrt{\mu_0 \rho_0}} \\ -\mathrm{i}\frac{k \ B_0 \cos \alpha}{\sqrt{\mu_0}} \sqrt{\rho_0} & -\mathrm{i}\frac{k \ B_0 \cos \alpha}{\mu_0} \end{pmatrix} \begin{pmatrix} \hat{v}_{1y} \\ \hat{B}_{1y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ .$$

This means that the vector solution has the form

$$\begin{pmatrix} \hat{v}_{1y} \\ \hat{B}_{1y} \end{pmatrix} = \begin{pmatrix} \nu \\ -\frac{\nu}{\sqrt{\mu_0 \rho_0}} \end{pmatrix} \ ,$$

where ν is a complex constant with dimensions of speed. So, we can see that the oscillations in B_{1y} are in exact antiphase to oscillations in v_{1y}, and decrease with decreasing density of plasma ρ_0.

Final Remarks

In this series we explored different systems which exhibit rich variety of oscillatory or waving behaviour, close to some local minimum of energy. The world of waves is not, however, limited to small oscillations, and different types of waves with similar characteristics (such as the conservation shape during propagation) exist, but some of their properties are different. For example, the equations that describe these waves may be non-linear, so the waves have a specific amplitude. Non-linear equations are, however, significantly more difficult to solve, mainly because we cannot apply the superposition principle. Such systems are an active area of research. However, the linear waves and oscillations still occur in an abundance of systems and we believe that the experience gained in this series will be valuable in your career in physics.

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