Problem IV.5 ... space visit

9 points; průměr 4,64; řešilo 55 studentů

Two aliens each live on their own space station. The stations are in free space and the distance between them is L. When one alien wants to visit the other, he has to board his non-relativistic rocket and fly to his neighbor. What is the shortest time an alien can spend on its way there and back? The mass of the rocket with fuel is m, without fuel m_0. The exhaust velocity is u. The fuel flow is arbitrary, and his neighbor won't let him load any fuel (he has little himself).

Jarda needed no one to notice that he had disappeared from a meeting for a while.

First, let us think about the optimization we need to perform. For an alien to spend the least possible amount of time on the route, he needs to fly as fast as possible the entire way. Therefore, the best option is to start accelerating very quickly right from the beginning to the desired speed v_1 and not waste time slowly accelerating. The volumetric flow rate of fuel is not limited, so we can imagine that the alien will instantly release some of his fuel with a velocity u. Similarly, he will not waste time braking, thus stopping immediately. We can apply these conditions to the return trip as well.

To solve the problem, we use the well-known Tsiolkovsky equation in the form

$$v_1 = u \ln \frac{m}{m_1}\,,$$

where v_1 is the speed at which the alien travels to his neighbor, and m_1 is the total mass of the rocket after it stops accelerating.

For the alien to come to a stand-still at his friend, he needs to slow down to zero, so the equation

$$v_1 = u \ln \frac{m_1}{m_2}\,,$$

must hold, where $m_1/m_2 = m/m_1$.

Similarly for the return trip $m_2/m_3 = m_3/m_0$.

We have two equations with three unknowns, and we choose m_2 as the remaining unknown (the rocket's mass when the alien stops at his friend). We express the time required to travel to the neighbor as a function of m_2

$$t_1 = \frac{L}{v_1} = \frac{L}{u \ln \frac{m_1}{m_2}} = \frac{L}{u \ln \frac{\sqrt{m m_2}}{m_2}} = \frac{2L}{u \ln \frac{m}{m_2}}\,.$$

Similarly,

$$t_2 = \frac{L}{v_2} = \frac{L}{u \ln \frac{m_2}{m_3}} = \frac{L}{u \ln \frac{m_2}{\sqrt{m m_2}}} = \frac{2L}{u \ln \frac{m_2}{m_0}}\,.$$

The total time is

$$T = t_1 + t_2 = \frac{2L}{u}\left(\frac{1}{\ln \frac{m_2}{m_0}} + \frac{1}{\ln \frac{m}{m_2}}\right) = \frac{2L}{u}\left(\frac{\ln \frac{m}{m_0}}{\ln \frac{m_2}{m_0} \ln \frac{m}{m_2}}\right)\,.$$

Now we differentiate with respect to m_2 and set the resulting function equal to zero, i.e.

$$\frac{\mathrm{d}T}{\mathrm{d}m_2} = -\frac{2L}{u} \ln \frac{m}{m_0} \frac{\left(\frac{1}{m_2} \ln \frac{m}{m_2} - \ln \frac{m_2}{m_0} \frac{1}{m_2}\right)}{\left(\ln \frac{m_2}{m_0} \ln \frac{m}{m_2}\right)^2} = 0\,,$$

and by simplifying, we finally get

$$m_2 = \sqrt{m m_0}\,.$$

It is evident that this point is the minimum of the function because if $m_2 \to m$, it would take a very long time to travel to the neighbor, while if $m_2 \to m_0$, the alien would spend a very long time on the return trip.

Substituting in the original function $T(m_2)$ gives us

$$T_{\mathrm{min}} = \frac{2L}{u}\left(\frac{\ln \frac{m}{m_0}}{\ln \sqrt{\frac{m}{m_0}} \ln \sqrt{\frac{m}{m_0}}}\right) = \frac{8L}{u}\frac{1}{\ln \frac{m}{m_0}}\,,$$

which is the result we were looking for.

Jaroslav Herman
jardah@fykos.org

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