Problem III.E ... acoustic thermometer 12 points; průměr 8,35; řešilo 49 studentů

Attach a string at two points at a fixed distance L and ensure it is always taut during measurement. Determine the dependence of the fundamental frequency of its oscillations on temperature. *Honza Benda is nuts.*

The main constraint from the problem statement is that the wire attachment distance must be constant. We can either construct this system or use a stringed instrument, such as a guitar. The theoretical framework of this problem aligns with the concepts explored in problem XXXVI.III.5[1], which investigates how the frequency produced by guitar strings varies with temperature. The relationship derived in that problem is given by

$$f = \sqrt{\frac{f_0^2 - \frac{E\alpha}{4L^2\rho}\Delta T}{1 + \alpha\Delta T}}\,,$$

where E denotes the Young's modulus, α the longitudinal thermal expansion coefficient, ρ the string density, L the string attachment distance, f_0 the string's original frequency and ΔT the string temperature difference.

The main observation is that a negative temperature change, i.e., cooling of the string, leads to a higher frequency, which we can expect. With fixed ends, the string tends to shorten, increasing the tension within the string and consequently raising the frequency. Conversely, heating the string results in a decrease in frequency.

Measurement procedure

We will use a classical six-string guitar, focusing on the thinnest E string. The distance between the nut and the bridge is $d = 65.0\,\mathrm{cm}$. We will change the temperature of the strings by placing them in an environment with a known temperature, and we will use a mobile phone with the Spectriod app for frequency measurements.

The strings are made of steel wires, with the G string and those below it wrapped with an additional bronze wire. An important parameter that varies for the strings is their diameter, starting with the thinnest string, where $d_B = 0.38\,\mathrm{mm}$, to the thickest with bronze winding, where $d_E = 1.32\,\mathrm{mm}$. However, for the wrapped strings, we only know the total thickness; we do not know if the thickness changes because of the steel wire or just the winding. Bronze and steel are both alloys, and their mechanical properties vary depending on the ratio of metals in them. Therefore, we consider these properties unknown and use them as free parameters in our measurements.

The initial measurements were taken at a temperature of $20.1\,^\circ\mathrm{C}$, at which we tuned the strings as close to their fundamental frequency as possible.

We always struck the string as hard as possible with a guitar pick to avoid heat transfer from the finger to the string. On repeated strikes, we measured the same frequency. When measuring with the Spectriod, we always see the full spectrum for a given time. The string emits not only the fundamental frequency but also its harmonic modes, which are integer multiples of the fundamental frequency. We have always chosen to measure only the fundamental frequency because it has the largest amplitude.

We then transferred the guitar to three other environments with different temperatures and let the whole guitar come into thermal equilibrium with the environment.

[1] `https://fykos.org/_media/year36/tasks/pdf/task36_3_5.pdf?cache=`

Tab. 1: Frequency of strings at 20.1 °C.

string	$\frac{f}{\mathrm{Hz}}$
B	246,6(1)
G	195,5(1)
D	146,6(1)
A	109,7(1)
E	81,90(1)

Tab. 2: Frequencies of the individual strings at different temperatures.

$\frac{t}{^\circ\mathrm{C}}$	$\frac{\Delta t}{^\circ\mathrm{C}}$	$\frac{f_\mathrm{B}}{\mathrm{Hz}}$	$\frac{f_\mathrm{G}}{\mathrm{Hz}}$	$\frac{f_\mathrm{D}}{\mathrm{Hz}}$	$\frac{f_\mathrm{A}}{\mathrm{Hz}}$	$\frac{f_\mathrm{E}}{\mathrm{Hz}}$
26,5(1)	6,4(2)	246,1(1)	195,1(1)	146,3(1)	109,5(1)	81,8(1)
20,1(1)	0,0(2)	246,6(1)	195,5(1)	146,6(1)	109,7(1)	81,9(1)
12,0(1)	-8,1(2)	247,7(1)	196,1(1)	146,9(1)	110,0(1)	82,1(1)
5,8(1)	-14,3(2)	248,5(1)	196,5(1)	147,1(1)	110,2(1)	82,3(1)

We would like to use the formula from the theory when fitting the data, but first, let us estimate the sizes of the individual terms. If we assume that the strings are composed of metals, we can expect the material values to be on the order of $\alpha \approx 10^{-5}\,\mathrm{K^{-1}}$, $E \approx 10^{11}\,\mathrm{Pa}$ and $\rho \approx \approx 10^4\,\mathrm{kg{\cdot}m^{-1}}$. Given the assumed magnitude of the thermal expansion and the temperature difference of about ten degrees Celsius, we can consider the denominator as unity. When fitting, we introduce a new parameter $\beta \equiv E\alpha/\rho$ to characterize the material properties of the string. The resulting theoretical dependence will be

$$f(x) = \sqrt{f_0^2 - \frac{\beta}{4L^2}x}\,.$$

A graph with the values and data fits is shown below, along with a table of fitting parameters.

Tab. 3: Fitting parameters.

string	B	G	D	A	E
f_0 [Hz]	246,76(8)	195,53(1)	146,57(2)	109,71(1)	81,93(2)
$\beta[\mathrm{Hz{\cdot}m^2{\cdot}K^{-1}}]$	99(7)	45(1)	19(1)	12,7(3)	6,7(7)

The graph and the measured data clearly show that the frequency change is less noticeable for thicker strings. That could be related to the additional bronze winding, which we did not account for in our theoretical model. The graph also demonstrates that the frequency changes due to temperature are small compared to the differences in the fundamental harmonic frequencies of the strings.

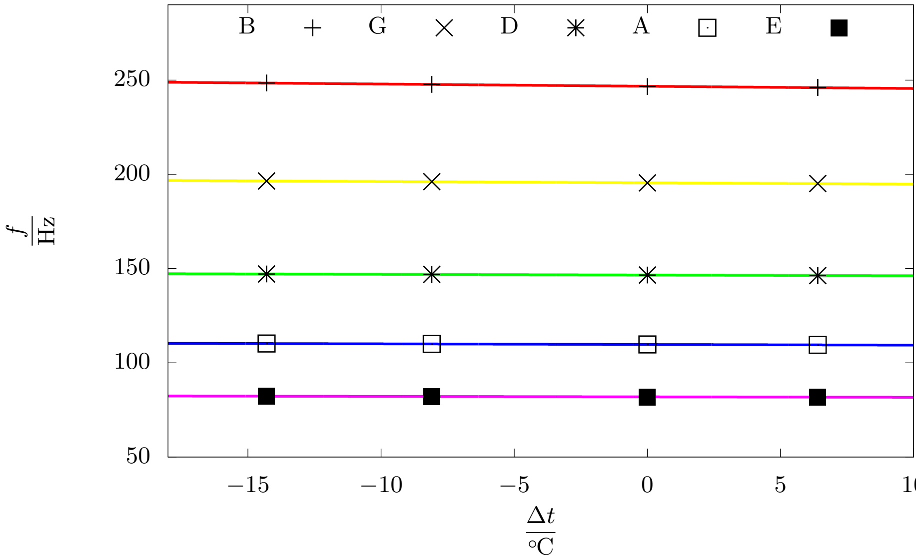

Fig. 1: Dependence of the string frequencies on the temperature change. Due to the small changes in frequency, the dependencies look almost constant.

Discussion

Several factors could have influenced our results given the chosen method of measurement. The process of heating or cooling the strings had the most significant impact. We could not change the temperature of the strings exclusively; instead, we allowed the whole guitar to reach the final temperature. This approach inevitably led to a change in the length of the guitar body. This effect could be overcome by electrically heating the strings. We would let an electric current flow through the string, and it would heat up through current heat losses. By connecting a thermistor, we would also determine the temperature of the string.

In the extreme case, where the guitar stretches more than the string, we might even observe a drop in the frequency of the string as the temperature decreases, which would reduce the tension in the string.

This effect was probably observed with decreasing material constant β. For string B, it corresponds to the usual value for metals $\beta_t = 10^{11} \cdot 10^{-5}/10^4\,\mathrm{Hz{\cdot}m^2{\cdot}K^{-1}} = 100$, but for the other strings, it decreases and the frequency differences are smaller. Other quantities, such as the cross-sectional area and density of the string, also experience temperature changes, which could have a noticeable effect. However, since these are area or volume quantities, the scaling effect is less significant. Additionally, the change in tension occurs only along the length of the string, so we do not expect significant changes in the transverse dimensions. The theoretical calculation can be found at the end of the problem referred to in the theory.

We could further consider the determination of the temperature of the strings. Because of the small heat capacity of the strings, we did not put the thermometer directly on the string to avoid temperature changes. That would not be realizable also because of their small surface area. Thus, we had to wait until the temperature of the guitar stabilized with its surroundings and assume that the strings were at the same temperature.

Furthermore, we did not consider a sudden change in the guitar's mounting. Big changes in tension can cause movement of the fastening pins in the bridge or the nut. We would observe this change as a significant change in the frequency of one string compared to the others.

Finally, using an instrument with longer strings, such as a bass or harp, could provide more noticeable changes. These instruments benefit from having a greater number of strings with different frequencies.

Results

We verified the trend of increasing frequency with decreasing temperature. We measured this dependence for five guitar strings, each characterized by a different β parameter, which reflects the material properties of the strings and may be influenced by factors such as the winding present in thicker strings.

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