Problem III.S ... weighted participants
10 points; průměr 5,43; řešilo 67 studentů

1. *According to definitions by International System of Units, convert these into base units*
   - *pressure* 1 psi,
   - *energy* 1 foot − pound,
   - *force* 1 dyn.
2. *In the diffraction experiment, table salt's grating constant (edge length of the elementary cell) was measured as* 563 pm. *We also know its density as* $2.16\,\mathrm{g{\cdot}cm^{-3}}$*, and that it crystallizes in a face-centered cubic lattice. Determine the value of the atomic mass unit.*
3. *A thin rod with a length l and a linear density λ lies on a cylinder with a radius R perpendicular to its axis of symmetry. A weight with mass m is placed at each end of the rod so that the rod is horizontal. We carefully increase the mass of one of the weights to M. What will be the angle between the rod and the horizontal direction? Assume that the rod does not slide off the cylinder.*
4. *How would you measure the mass of:*
   - *an astronaut on ISS,*
   - *a loaded oil tanker,*
   - *a small asteroid heading towards Earth?*

Dodo keeps confusing weight nad mass.

Problem 1

The unit psi is an abbreviation of the English expression "pound per square inch", more precisely "pound-force per square inch". The units thus correspond to the definition of pressure as the proportion of force applied to an area. A pound of force is a unit of force that expresses the gravitational force exerted by a mass of one pound. This is now defined exactly as 0.453 592 37 kg and multiplying by the normal acceleration of gravity $9.806\,65\,\mathrm{m{\cdot}s^{-2}}$ we get the pound of force as 4.448 22 N.

An inch has a length of exactly $2.54\,\mathrm{cm} = 0.025\,4\,\mathrm{m}$, a square inch thus equates to an area of $0.025\,4^2\,\mathrm{m^2} = 6.451\,6\cdot 10^{-4}\,\mathrm{m^2}$. This means, that the unit psi represented in SI units is $1\,\mathrm{psi} = 4.448\,22\,\mathrm{N}/6.451\,6\cdot 10^{-4}\,\mathrm{m^2} = 6\,894.75\,\mathrm{Pa}$. This unit is commonly used, for example, with various valves.

The unit foot − pound = ft·lb expresses the work done by one pound of force in moving a body one foot. Again, this unit represents the natural introduction of a quantity, similar to psi. We already know that one pound of force is 4.448 22 N, so we just need to multiply it by one foot, which is exactly 0.304 8 m, to get $1\,\mathrm{ft{\cdot}lb} = 1.355\,8\,\mathrm{J}$. We can see that this unit is similar to one joule, so we don't need to convert anything for order-of-magnitude energy estimates. It is also worth noting that there is also a unit of "pound-foot", which is a unit of moment of force (similar to our N·m). Its SI value is, of course, the same as that of "foot-pound".

The previous two units studied belonged to the imperial unit system and are now more commonly used only in the USA, Canada, and the UK. In contrast, the unit "dyne" belongs to the CGS system, where the basic units are the centimeter, gram, and second. This system was the predecessor of today's SI. Thus, the definition of a dyne is the force required to accelerate a body of mass 1 g by $1\,\mathrm{cm{\cdot}s^{-2}}$. Substituting SI units into this definition, we get $1\,\mathrm{dyne} = 10^{-3}\,\mathrm{kg}\cdot 10^{-2}\,\mathrm{m{\cdot}s^{-2}} = 10^{-5}\,\mathrm{N}$.

Problem 2

In solving the problem we calculate the mass of one elementary cell, the mass of individual atoms and by comparing them we find the atomic mass unit we are looking for.

First, let's recall what an elementary cell of rock salt, or NaCl, looks like. Consider first the sodium atoms (Na), which are located in the corners of the cube with side a and in the centers of its walls. This corresponds to a cubic area-centered lattice with a lattice constant of just $a = 563\,\mathrm{pm}$. We now add its partner chlorine (Cl) to one sodium atom so that it is displaced from sodium by $a/2$ along the edge of the elementary cell. Repeat for all the sodium atoms in the crystal – put chlorine next to them. So it's clear that the ratio of sodium to chlorine is one, just as we can see from the chemical formula. The sodium atoms form a cubic area-centered lattice, as do the chlorine atoms, but this lattice is tightly displaced.[1] We might ask why we do not choose as the elementary cell a cube with half the edge length and with sodium and chlorine atoms at the corners. This is since the translational symmetry in the crystal would be broken. It simply expresses that if we translate by one length a along the cube's edge, we get the same surroundings as before the translation. However, in the case of the smaller cube under consideration, we would move from a sodium atom to a chlorine atom, for example, the surroundings would change.

In a cubic area-centred lattice there are 4 atoms per cell. An atom in each corner is shared by eight identical cells, but again each cube has eight vertices, so there is just one corner atom per cell. With the atoms in the centers of the walls, it is similar, each one is shared by two cells, but there are six walls in total, which equates to three wall atoms per cell. Since in our cell, one sodium atom has one chlorine atom attached to it, there are 4 sodium atoms and 4 chlorine atoms in a cube of edge a.

Now, finally, we can move on. The relative atomic mass of sodium is $A_{\mathrm{Na}} = 22.99$, so one sodium atom has an average mass of 22.99 atomic mass units. Similarly, for chlorine, $A_{\mathrm{Cl}} = 35.45$. The mass of one elementary cell is therefore

$$m_{\square} = 4A_{\mathrm{Na}} \cdot m_{\mathrm{u}} + 4A_{\mathrm{Cl}} \cdot m_{\mathrm{u}} = 233.8 \cdot m_{\mathrm{u}}\,.$$

From the knowledge of the density and volume of an elementary cell, we can calculate its weight as

$$m_{\square} = \rho V = \rho a^3 = 2\,160\,\mathrm{kg{\cdot}m^{-3}} \cdot \left(563 \cdot 10^{-12}\,\mathrm{m}\right)^3 = 3.85 \cdot 10^{-25}\,\mathrm{kg}\,.$$

From the equality of both expressions of $m_{\square}$ we get

$$m_{\mathrm{u}} = \frac{\rho a^3}{233.8} = 1.65 \cdot 10^{-27}\,\mathrm{kg}\,,$$

which is quite close to the stated value of $m_{\mathrm{u}} = 1.660\,5 \cdot 10^{-27}\,\mathrm{kg}$.

Problem 3

Consider a situation where the rod has deflected by an angle φ measured from the original point of contact to the current one. On one side, after the deflection, the rod is $l/2 + \varphi R$ long, where φ is the angle of the point of contact of the rod with the cylinder measured from the perpendicular and given in radians. On the other side, its length is $l/2 - \varphi R$. The rod did not

[1] `https://upload.wikimedia.org/wikipedia/commons/f/f9/Ionlattice-fcc.svg`

move during deflection, so the point of contact must have traveled the same distance as the angle φ on the cylinder.

Consider the horizontal projections of the distances of the centers of gravity of the parts (i.e. the two weights and the parts of the bar) from the point of contact. We multiply these distances by the weight exerted by each part to obtain the moments of the forces. At one end is the torque

$$M_1 = \left(Mg\left(\frac{l}{2} - \varphi R\right) + \lambda\left(\frac{l}{2} - \varphi R\right) g \frac{\frac{l}{2} - \varphi R}{2}\right) \cos\varphi\,,$$

and at the other end

$$M_2 = \left(mg\left(\frac{l}{2} + \varphi R\right) + \lambda\left(\frac{l}{2} + \varphi R\right) g \frac{\frac{l}{2} + \varphi R}{2}\right) \cos\varphi\,.$$

The mass of a part of the bar is determined as the simple product of its length and length density; the center of gravity is located in its middle. We apply this reasoning separately for each of the parts of the bar.

The rod no longer moves, so the equality of the torques gives the relation

$$Ml - 2M\varphi R + \lambda\left(\frac{l^2}{4} - l\varphi R + \varphi^2 R^2\right) = ml + 2m\varphi R + \lambda\left(\frac{l^2}{4} + l\varphi R + \varphi^2 R^2\right)\,.$$

$$Ml - 2M\varphi R - \lambda l\varphi R = ml + 2m\varphi R + \lambda l \varphi R\,.$$

from which we get

$$\varphi = \frac{l}{2R}\frac{(M-m)}{(m+M) + \lambda l}\,.$$

We see that the offset is proportional to the added mass $M - m$. By using the tilt angle, we can see how much mass we added. For practical use, we would need to measure the angle accurately and not overload the rod so that it doesn't slide off the cylinder or tip over.

Problem 4

Since gravity balances the centrifugal force in orbit, astronauts move in weightlessness. They cannot use the basic principle of weighting that we use on Earth, which is to convert gravity into mass using the value of the acceleration caused by gravity. It is impossible to use a force of the form $F = mg$, you need to use a force of $F = ma$ to determine the mass, so the astronaut has to change his momentum and his mass is determined accordingly.

A simple example that can be implemented in these conditions is a spring harmonic oscillator. Its balance of forces is $-ky = ma$, where k is the spring stiffness and y is the deflection from the equilibrium position. The solution to this equation is harmonic oscillations with period

$$T = 2\pi\sqrt{\frac{m}{k}}\,.$$

By knowing the stiffness of the spring and measuring the period of oscillation, we can determine the mass of bodies even in weightlessness and with limited space. Astronauts do have equipment based on this principle on the ISS.

To determine the mass of an oil tanker, we use Archimedes' law. The buoyant force of the water must balance the gravitational force exerted by the entire tanker in the equation

$$mg = V\rho_\mathrm{w} g \quad \Rightarrow \quad m = V\rho_\mathrm{w}\,,$$

where m is the mass, g is the gravitational acceleration, V is the volume of the submerged part of the ship, and $V\rho_\mathrm{w}$ is the density of water. We can hopefully measure the surrounding water's density and determine the volume V of the submerged part by knowing the ship's profile and depth of submergence. Ships have grooves on them by which the depth of the dive can be determined. The ship's shape below the surface may be such that we can't directly express its volume using some formula, but if we know what the ship looks like, it's certainly not a problem to calculate it. Therefore, the ship's crew has a conversion from the measured depth of submergence to the submerged volume.

For asteroids, we use yet another way to measure their mass – we look at how gravity affects objects in their vicinity. We track the position of both the asteroid and the other object, and we can calculate the mass from, for example, the orbital period of both bodies. This is an example of a situation where an asteroid has its moon. However, such situations are rare since asteroids usually do not have enough mass to capture and hold an object in their gravitational field. If such a case occurs, the moonlets may be so small that it is impossible to detect them at all.

If two asteroids come close together, we can determine their mass from the deviation of their mutual motion. Ideally, one of the bodies is a probe. However, this solution tends to be very costly.

If there is no body on which the studied asteroid can exert a gravitational influence, we have no choice but to rely on an estimate. From the infrared brightness, we determine the asteroid's dimensions and estimate the density based on experience from other measurements. The calculated mass is then more of an order-of-magnitude approximation.

Commentary on submitted solutions

The first subproblem was for 2 points, and only those who calculated the result from the definition i.e. wrote down e.g. the weight of a pound or the length of a foot, could get the full number of points. If someone wrote only the correct result, they were only credited one point.

The second subproblem was also worth 2 points. Here you mostly correctly calculated the mass of one elementary cell, but the problem was to determine how many and what atoms were in it and what their relative masses were. Only those with the correct procedure and result (which was not difficult to check in any tables) got full points.

In the third part, we gave 3 points. The first point was for stating the equality of forces, the next for stating the relationship between the position of the center of the rod from the point of contact with the cylinder and the angle of deflection, and for some basic calculation. The full number of points was then for the correct solution. Unfortunately, the assignment was not very clear and many of you imagined the situation differently than we intended. In this case, we scored the correct calculation when the problem was incorrectly sketched.

In the last part, each question was worth one point. For the tanker, most of the answers related to the buoyant force were considered correct. For weighing the astronaut, a common answer was to apply a force to the astronaut and measure the acceleration. Such an answer was not recognized unless it was specified what force to apply and how to apply it. On Earth, when weighing, we also apply a force – gravity, but that is not available in space and needs to be replaced in some specific way. For the asteroid problem, it was necessary to realize that its mass could not be determined from its trajectory in the gravitational field of a significantly more massive body. Answers that considered the question of collision with a body of known mass, similar to the astronaut problem, were considered correct.

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