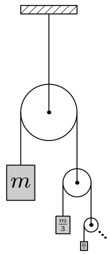
Problem VI.4 ... infinite pulleys

7 points; (chybí statistiky)

Let us have an infinite system of intangible pulleys as shown in the figure, where the mass of each additional weight is one-third of the weight of the previous one. What is the acceleration of the first weight of mass m? *Matěj was looking for the difference between countably and uncountably many pulleys.*



Substitution introduction

Before we go into the infinite pulleys, let's derive a rule on how to simplify the pulley scheme, which is analogous to the problem of serial vs. parallel resistors in a circuit. Here, we will have just one rule.

Let us first consider the case of a single simple pulley, which is attached to some string on which it exerts a force of F_0. There is one weight on each side of the pulley. Let us denote them by m_1 and m_2. We further denote the acceleration acting on the pulley by a_0. (if the pulley is suspended rigidly, then $a_0 = g$, due to the non-inertial system, but this acceleration may be different).

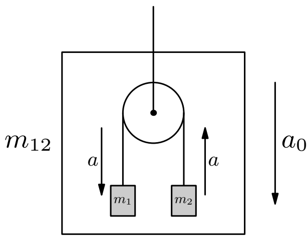


Fig. 1: Nákres situace.

On a system of weights m_1 and m_2 connected by a string, the forces m_1a_0 and m_2a_0 act in opposite directions and because both weights must move with the same acceleration a relative to the pulley according to Newton's second law, we can write

$$(m_1 + m_2)a = m_1a_0 - m_2a_0,$$

$$a = \frac{m_1 - m_2}{m_1 + m_2} a_0, \tag{1}$$

where we have set a positive direction a for the downward motion of the weight m_1 and m_2 upward (relative to the pulley).

We express the force by which the string is stretched by the acceleration of the first weight

$$F = m_1(a_0 - a).$$

By the way, we would get the same result for the second weight. We plug in for a

$$F = m_1 \left(1 - \frac{m_1 - m_2}{m_1 + m_2}\right) a_0 = \frac{2m_1m_2}{m_1 + m_2} a_0.$$

The force that stretches the string suspending the pulley is doubled

$$F_0 = \frac{4m_1m_2}{m_1+m_2}a_0. \tag{2}$$

We see that this force is proportional to the acceleration of the pulley. This means that we can replace the entire pulley with one weight of mass

$$m_{12} = \frac{4m_1m_2}{m_1+m_2}, \tag{3}$$

which does not change the net force on the top string.

Infinity Replacement

Let us denote the coefficient of mass decay in the system $k = 1/3$ and solve the whole problem for the general case where k can be any positive number.

Our infinite system of pulleys suspended by a single string will also be replaced by some single weight whose force effects on the string are the same. Let us denote the as-yet-unknown mass of such a weight by M.

Now, let us take the first (largest) pulley with the first weight. This leaves us with the same infinite system as last time, only all the masses will be k times different. This means we can replace this system with one weight with a mass of kM (see figure 2).

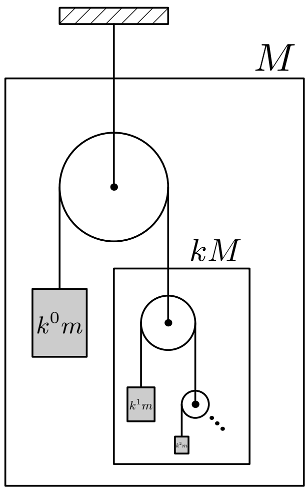


Fig. 2: Nákres situace.

So we have one pulley with weights m and kM. At the same time, we know that when these two weights are replaced by one weight, according to the (3), we must get mass M

$$M = \frac{4mkM}{m+kM}\,.$$

This equation has two solutions for M:

$$M_1 = 0\,,$$
$$M_2 = \frac{(4k-1)m}{k}\,.$$

Solution I (non-physical)

For the first case $M_1 = 0$, the following emerges

$$a_1 = g\,.$$

In this case, all weights fall freely. We can consider it the case where we have a finite (but arbitrarily large) number of pulleys. Thus, there is a last pulley such that no smaller pulley is suspended below. This string, attached to the last weight in our series, must not be loaded with any counterweight (see (2) for the case of one weight equal to zero $F_0 = 0$). Thus, each additional pulley placed above will also be replaced by a zero mass, and no weight will thus slow any of the higher weights, so the whole system will free fall.

This idea is only valid in the idealized case where all strings and pulleys are massless. But then, it is quite understandable that all the weights will fall to the ground freely because there is always zero mass on the other side of their pulleys.

Solution II (more realistic)

For the second case, we have one pulley with weights m and $kM = (4k-1)m$. Acceleration of the first weight is calculated using the relation (1) (where we plug both masses a $a_0 = g$)

$$a = \frac{m-(4k-1)m}{m+(4k-1)m}g\,,$$

$$a = \frac{1-2k}{2k}g\,.$$

Let's see what this result tells us. If $k = 1/2$, the whole system will be in equilibrium, and the acceleration will be zero. For larger k the acceleration will be upward, and the magnitude will be limitingly close to k $-g$. For $k < 1/2$ the first weight will accelerate downward, and the limiting acceleration will approach k g.

For our particular case $k = 1/3$ it follows as this

$$a = \frac{1}{2}g\,.$$

Conclusion

In a real situation, we'll never have ∞ pulleys. In physics, ∞ is often used only as a mathematical tool to describe large quantities of objects or events because it makes calculations much simpler than if we would have to calculate with a finite number of objects. When a physicist says that some quantity X is infinite, he typically means that in all calculations (formally) he uses the limit $\lim_{X\to\infty}$. In most cases, when we have infinite quantities in computation, it doesn't matter which limit we make first. However, there are examples in mathematics where it indeed depends on the order of limits. As an example

$$\lim_{x\to 1^-} \lim_{n\to\infty} x^n = \lim_{x\to 1^-} 0 = 0$$
$$\lim_{n\to\infty} \lim_{x\to 1^-} x^n = \lim_{n\to\infty} 1^n = 1$$

We would run into something similar if we were to solve our pulley problem rigorously. The formal solution would be to solve the example for one pulley, then for two pulleys, and so on. We would need to find a general relationship between the acceleration of the first pulley and the number of pulleys involved in our system. Finally, we would send $n \to \infty$, and we would get the correct result. But the problem is that a system of n pulleys connected in succession has no clear solution because we don't know the tension in the last string. If the tension is zero (there is no counterweight), obviously all n pulleys will free fall. However, if we consider that the last string does not have zero tension, the pulley system will move in a non-trivial way that could be accurately calculated. Thus, this is a similar problem to the x^n limit order problem. If we add pulleys such that the last string is free, we get the solution I, even if we use ∞ pulleys. On the other hand, if there is any non-zero tension in the last string, the acceleration of the first pulley will gradually converge to the solution. When we would add an infinity of pulleys in succession, the solution II becomes exact.

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